



A Ritz's method based solution for the contact problem of a deformable rectangular plate on an elastic quarter-space

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ABSTRACT

In this article, Ritz's method is used to calculate with unprecedented accuracy the displacements related to a deformable rectangular plate resting on the surface of an elastic quarter-space. To achieve this required three basic steps. The first step involved the study of Green's function describing the vertical displacements of the surface of an elastic quarter-space due to vertical force applied on its surface. For this case, an explicit formula was obtained by analytically resolving a complicated integral that did not previously have an analytical solution. The second step involved the study of the coupled system of a plate and an elastic quarter-space. This portion focused on determining reactive forces in the contact zone based on Hetenyi's solution. After determination of the reactive forces, certain features were attributed to the plate's edges. The final step involved the application of Ritz's method to determine the deflections of the plate resting on the surface of the quarter-space. Finally, an example calculation and validation of results are given. This is the first semi-analytical solution proposed for this type of contact problem.

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1. Introduction

The contact problem of a deformable rectangular plate resting on the surface of an elastic foundation has not had an exact solution to the present time. Despite its significant importance for many geotechnical problems (amongst other fields), even a simple, locally-deformable elastic foundation (Winkler's model) has no exact solution (Palatnikov, 1964). The main difficulty in finding exact solutions for contact problem is in satisfying the static boundary conditions along the plate's edges. The problem is further complicated when the considered model is other than the two classical and most known ones, namely Winkler's model and Boussinesq's model (half-space). Neither address the specific features related to the quarter-space's model. Other related studies of this model are very limited at the present time. However, the following points were raised by some researchers.

Boudjelkha and Diaz (1972) obtained the solution for the quarter-space Dirichlet problem for Laplace's equation based on the Poisson integral formula for that of the half-space. This enabled determination of a function to specify the partial differentiation equation in the interior of a given region that prescribed values on the boundary of the region. They deduced a representation theorem for harmonic functions in the quarter-space.

In a related study, Hanson and Keer (1990) analyzed the problem of determination of the elastic stress and displacements in a quarter-space under arbitrarily applied surface loadings, for which they developed a solution that may then be used to analyze contact problems for an elastic quarter-space based upon on a special solution for a half-space, which isolates the singularity, and is incorporated into the numerical-boundary-element solution of the integral equations. Once the equations are solved, the solution for elastic quarter-space can be found. The principle of quarter-space is also used in the science of the cosmos. For example, Schultz and Finn-Foley (2007) described the uses, advantages, and drawbacks of quarter-space experiments compared to half-space experiments when examining impact dynamics. The numerical solution for an absolutely rigid rectangular plate resting on an elastic quarter-space was solved previously by Aleksandrov and Pozharskogo (1998).

Rigorous mathematical analyses have also been applied to the study of deformable plates resting on an elastic foundation. Tseytlin (1984) proposed a solution for the axi-symmetric bending of circular plates resting on an elastic foundation by using the plate's vertical displacements, in the form of Eigen functions of a differential operator of the axi-symmetric vibrations of a circular plate with free edges. Later, this idea was implemented in the solution of contact problems for rod and ring plates (Bosakov, 2006). Attempts to derive similar relationships for a bending rectangular plates resting on elastic foundation with free edges have

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previously failed to such an extent that pessimistic predictions about the viability of obtaining Eigen functions of the flexural vibrations of a rectangular plate resting on an elastic foundation with free edges have been published (Zecai, 1988).

On a related subject, in 1908, Ritz considered the problem of vibrations of a rectangular plate with free edges (Timoshenko, 1959). The normal function of vibration of a rod was taken to express the coordinate functions of the vibration of plate. Later, Kontorovich and Krylov (1962) also used these functions to solve the rectangular plate vibration problem. Korenev and Rabinovich (1972) and Bolotin (1978) then used these coordinate functions for calculation of rectangular plates resting on an elastic foundation, but they were only able to generate an approximate solution for this problem.

To more accurately depict the deflection of a deformable rectangular plate resting on the surface of an elastic quarter-space, a coordinate function of deflections using a quasi-Eigen function of a differential operator of the flexural vibrations of a rectangular plate with free edges was derived by Bosakov (2007).

The problem has been classified as belonging to the order of non-classical contact problems (Aleksandrov et al., 1976). Several approximate calculation methods exist to determine reactive forces and/or displacement for a rectangular plate resting on the surface of an elastic foundation. Prominent amongst these are the double power series by Gorbunov-Posad et al. (1984), the Zhemochkin method (Zhemochkin and Sinitsyn, 1962), and other numerical approaches, such as the boundary element method Aliabadi (2002), and the finite element method Hild and Laborde (2002).

The problem of a flexible surface exerting stresses along another flexible surface regularly occurs in Geotechnics, where Winkler's model is most commonly applied. The approach has been adopted most frequently for soil–structure interaction for shallow foundations (Massalas et al., 1978; Kubenko et al., 2006), but used as well for laterally loaded deep foundations (Horvath, 1984), retaining walls (Laefer (2001)), mining (Pytel and Chugh, 1989), tunneling (Najm and Ishijima, 1993), roadway repair (Cho et al., 1996), runway performance prediction (Pan and Atluri, 1995), train–railway track interaction (Knothe and Le-The, 1985), silo construction (Vasilenko and Emelyanov, 1993), and soil–structure–water interaction problems (Stefanou, 1983). Additionally, similar concerns arise in a variety of materials either in their manufacturing or performance, particularly when thin coatings are overlain on other substrates (Chai, 2003; Sriram et al., 2003; Wepelmann and Swain, 1996). Examples include laminated glass (Grant et al., 1998), austenitic stainless steel (Stellwag, 1998), and dental work (Huang et al., 2005).

2. Problem scope

In this paper a deformable rectangular plate resting on the surface of an elastic quarter-space with free edges is considered, with the goal of determining a solution very close to the exact one, when subjected to external loading. A version of this problem is shown in Fig. 1 with an external, concentrated, point load in the center, but the scope of this work is applicable to distributed external loads, as well. The problem is to determine the distribution of reactive forces in the contact zone, between the plate and the surface of quarter-space, as well as the vertical displacements of the plate. No shear stresses are considered in the contact area. Below, a solution of the bending rectangular plate problem based on the Ritz method (Timoshenko, 1959) is presented.

3. Determination of the vertical displacements of the surface of quarter-space

First, an analytic expression for the vertical displacements of the surface of an elastic quarter-space subjected to a concentrated

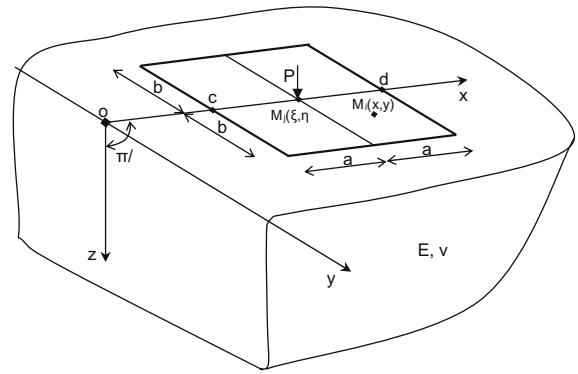


Fig. 1. Rectangular plate resting on the surface of an elastic quarter-space.

vertical force P was obtained. In the literature, such a task is called Hetenyi's solution, (Hetenyi, 1970). Keer et al. (1983) first derived the solution for a quarter-space using the principle of superposition of two half-spaces subjected to a symmetrically concentrated vertical force.

In the work of Bosakov et al. (2001) [by extrapolation the approach of Uflyand (1972)], an exact expression for the unknown displacements V of the surface of quarter-space in the form of triple integral was obtained as shown in (1):

$$V = \frac{P(1 - \nu)}{2G\pi^3} \int_0^\infty \int_0^\infty \int_0^\infty K(t, \tau) K_{it}(\sigma x) \cos(\tau t) \times \cos(\sigma y) e^{-\sigma u \cosh(t)} dt d\tau d\sigma \tag{1}$$

where

$$K(t, \tau) = \frac{[\sinh(\pi\tau)]^2}{[\sinh(\pi\tau)]^2 - \left(\tau - 2\varepsilon \frac{\tanh(\pi\tau/4)}{\cosh(2t)}\right)^2} \tag{2}$$

and $G = \frac{E}{2(1+\nu)}$, $\varepsilon = 1 - 2\nu$, E and ν are modulus and Poisson's ratio of the quarter-space; P is the external force; t, τ, σ are variables of integration; u the distance between the origin and the point where the force is applied (Fig. 2); $K_{it}(\sigma x)$ the MacDonal function with argument (σx) , where $i = \sqrt{-1}$ (Gradshteyn and Ryzhik, 1969).

Using a special method of approximation, based on the work of Vorovich et al. (1974) a convenient formula for the computation across elementary functions was obtained from (1). For this purpose, the function $K(t, \tau)$ was expanded over the small parameter $\varepsilon = 1 - 2\nu$ and restricted to the first two members of series. From that Eqs. (3)–(5) were obtained:

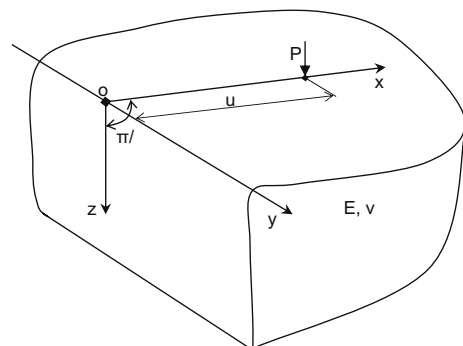


Fig. 2. A concentrated force on the elastic quarter-space's surface.

$$K(t, \tau) \approx L_1(\tau) - \frac{4\varepsilon}{\cosh(2t)} L_2(\tau) \tag{3}$$

$$L_1(\tau) = \frac{2[\sinh(\pi\tau)]^2}{\cosh(\pi\tau) - 1 - 2\tau^2} \tag{4}$$

$$L_2(\tau) = \frac{\tau[\sinh(\pi\tau)]^2 \tanh(\pi\tau/4)}{([\sinh(\pi\tau/2)]^2 - \tau^2)^2} \tag{5}$$

Comparison of the graphs of functions $K(t, \tau)$ at $\nu = 1/3$, with the formulas (2) and (3), as depicted in Fig. 3, shows an excellent agreement between them.

Next, the asymptotic properties of $\frac{1}{\cosh(\pi\tau)} L_1(\tau)$ and $L_2(\tau)$ are considered with respect to (6) below:

$$\begin{aligned} \lim_{\tau \rightarrow 0} L_1(\tau) \frac{1}{\cosh(\pi\tau)} &= \frac{4\pi^2}{\pi^2 - 4} \\ \lim_{\tau \rightarrow \infty} L_1(\tau) \frac{1}{\cosh(\pi\tau)} &= 2 \\ \lim_{\tau \rightarrow 0} L_2(\tau) &= \frac{4\pi^3}{(\pi^2 - 4)^2} \\ \lim_{\tau \rightarrow \infty} L_2(\tau) &= 4\tau \end{aligned} \tag{6}$$

In accordance with the asymptotic properties in (6), the term $L_1(\tau)$ was approximated over the interval $[0, \infty[$ by the expression (8)

$$L_1(\tau) = \left[2 + \frac{2\pi^2 + 8}{(\pi^2 - 4) \cosh(\pi\tau)} \right] \cosh(\pi\tau) \tag{7}$$

Fig. 3 is also nearly identical to the function $L_1(\tau)$, which is given by (4) and (7).

In the same way, an approximation was made over the interval $[0, \infty[$ the function $L_2(\tau)$ by following expression (8):

$$L_2(\tau) = 4\tau \coth\left(\frac{(\pi^2 - 4)^2 \tau}{\pi^3}\right) \tag{8}$$

Fig. 4 illustrates the function $L_2(\tau)$, which is given by (5) and (8).

However, after considerable mathematical difficulties the integral: $I = \int_0^\infty \tau \coth\left(\frac{(\pi^2 - 4)^2 \tau}{\pi^3}\right) \cos(\tau t) K_{it}(\sigma x) d\tau$, included in the formula (3) of $L_2(\tau)$ still defied exact calculation. Therefore, by determining displacements $V(x, y)$, $L_2(\tau)$ could be decomposed in a power series in the neighborhood of $\tau = 0$ as shown in (9):

$$\begin{aligned} L_2(\tau) \approx & 3.59991 + 1.14597\tau^2 - 0.00203\tau^4 - 0.00997\tau^6 \\ & + 0.00141\tau^8 - 0.000064\tau^{10} + \dots \end{aligned} \tag{9}$$

from which only the first few terms were retained.

Neglect of the remaining members of the series (9) only insignificantly affects the value of the final displacements $V(x, y)$. In support of this, Fig. 5 shows displacements $V(x, y)$, constructed by taking into account 2 and 3 members of the series (9) at $y = 0$.

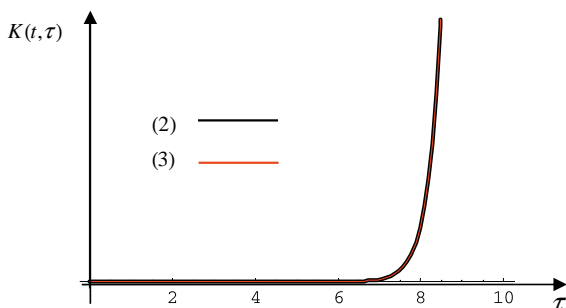


Fig. 3. Graphical comparison of the function $K(t, \tau)$ by formulas (2) and (3); nearly identical in appearance to the function $L_1(\tau)$ by formulas (4) and (7).

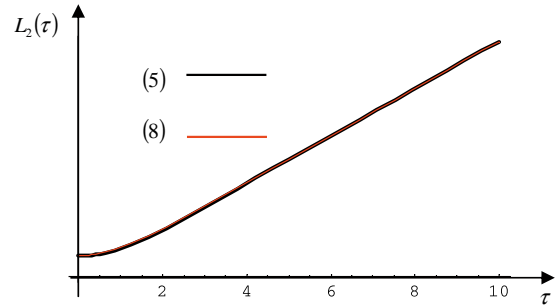


Fig. 4. Graphics comparison of the function $L_2(\tau)$ by formulas (5) and (8).

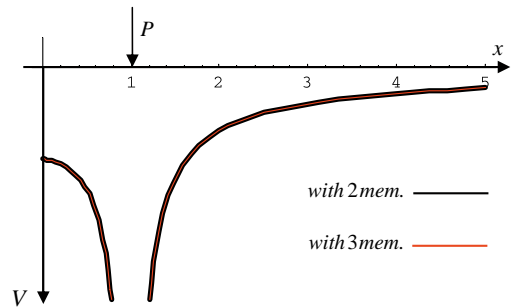


Fig. 5. Effect of the terms number of series (9) on the value of the final displacements.

Next, the values of the integrals in (10) as previously derived by Gradshteyn and Ryzhik (1969), Rektoris (1985) were used to calculate the integral expression (1).

$$\begin{cases} \int_0^\infty \cos(\tau t) e^{-\sigma u \cosh(\tau)} d\tau = K_{it}(\sigma u) \\ \int_0^\infty K_{it}(\sigma x) K_{it}(\sigma u) \cos(\sigma y) d\sigma = \frac{\pi^2}{4\sqrt{ux} \cosh(\pi\tau)} P_{\frac{1}{2}+it}(\cosh(\mu)) \\ \int_0^\infty P_{\frac{1}{2}+it}(\cosh(\mu)) \frac{\cos(\beta\tau)}{\cosh(\pi\tau)} d\tau = \frac{1}{\sqrt{2} \cosh(\mu) + \cos(\beta)} \\ \int_0^\infty \cos(\tau t) K_{it}(\sigma x) d\tau = \frac{\pi}{2} e^{-\sigma x \cosh(\tau)} \end{cases} \tag{10}$$

where $P_{\frac{1}{2}+it}(\cosh(\mu))$ is the Legendre function (Gradshteyn and Ryzhik, 1969); β takes 0 or π ; $\cosh(\mu) = \frac{u^2 + x^2 + y^2}{2ux}$.

Omitting intermediate calculations, the required expression for the quarter-space's surface displacements caused by the action of a concentrated force P , is expressed in (11) in terms of elementary functions with the first three terms of series (9):

$$\begin{aligned} V(x, y) = & \frac{P(1 - \nu^2)}{\pi E} \left\{ \frac{1}{R_1} + B_0 \frac{1}{R_2} - (1 - 2\nu) \left[B_1 \frac{(u+x) \left(\sqrt{2} - \frac{u+x}{R_1} \right)}{y^2 + R_2^2} \right. \right. \\ & \left. \left. - B_2 \frac{ux}{R_2^2} - B_3 \pi \frac{ux(u^2 - 7ux + x^2 + y^2)}{2R_2^5} \right] \right\} \end{aligned} \tag{11}$$

$$R_1 = \sqrt{(u-x)^2 + y^2}; \quad R_2 = \sqrt{(u+x)^2 + y^2}; \quad B_0 = \frac{\pi^2 - 4}{\pi^2}; \quad B_1 = 1.59991; \quad B_2 = 1.14597; \quad B_3 = 0.00812.$$

Of note is that the first term in (11) corresponds to the Boussinesq solution for an elastic half-space (Gorbunov-Posad et al., 1984) and contains a singularity. The remaining terms are smooth continuous functions with decay at infinity as $1/x$. On the edge of the wedge ($x = 0$), displacement is limited.

Fig. 6 shows that the vertical displacements of the quarter-space's surface depends on $\frac{P(1-\nu^2)}{\pi E a}$ with $\nu = 1/3$, due to the concentrated force P situated at a distance ($u = 1.5$ m) from the edge.

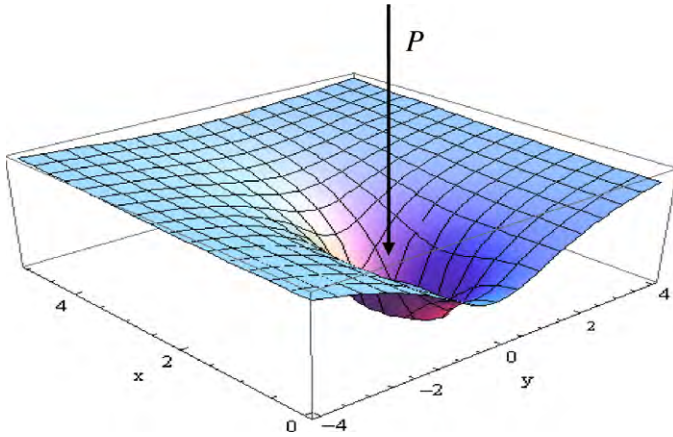


Fig. 6. Vertical displacements of the quarter-space's surface due to a concentrated force.

4. Deflections of a rectangular plate resting on the surface of an elastic quarter-space

The deflection of a rectangular plate $W(x, y)$ with dimensions $(2a \times 2b)$, as resting on an elastic quarter-space's surface (Fig. 1) has been represented previously Bosakov (2007) as (12):

$$W(x, y) = A_0 + A_1 \frac{x}{a} + A_2 W_2(x, y) \tag{12}$$

where A_0, A_1, A_2 are undetermined coefficients; $W_2(x, y)$ is the quasi-Eigen function of the differential operator of flexural vibrations of a rectangular plate with free edges, as obtained by Bosakov (2007):

$$W_2(x, y) = \cos\left(\alpha_2 \frac{x}{a}\right) \cos\left(\beta_2 \frac{y}{b}\right) - \frac{\sin(\alpha_2) \sin(\beta_2)}{\sinh(\alpha_2) \sinh(\beta_2)} \cosh\left(\alpha_2 \frac{x}{a}\right) \times \cosh\left(\beta_2 \frac{y}{b}\right), \quad \alpha_2 = \beta_2 = 2.36502. \tag{13}$$

The first term in (12) represents the vertical displacement of the plate and the second term represents its rotation.

5. Solution

5.1. Method

The distribution of the reactive forces in the contact zone between the rectangular plate and the quarter-space's surface was assumed to be (14)

$$\rho(x, y) = \frac{1}{\sqrt{1 - \frac{x^2}{a^2}} \sqrt{1 - \frac{y^2}{b^2}}} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} B_{ik} T_i\left(\frac{x}{a}\right) T_k\left(\frac{y}{b}\right) \tag{14}$$

where $T_i(\frac{x}{a}), T_k(\frac{y}{b})$ based on Chebyshev polynomials (Gradshteyn and Ryzhik, 1969); B_{ik} are undetermined coefficients.

Hetenyi's solution, (Hetenyi, 1970), for the displacement of a point $M_i(x, y)$, located on the quarter-space's surface due to the unit force at a point $M_j(\xi, \eta)$ can be represented as (15)

$$V(x, y, \xi, \eta) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} C_{mn}^{pq}(\alpha) T_m\left(\frac{x}{a}\right) T_n\left(\frac{y}{b}\right) T_p\left(\frac{\xi}{a}\right) T_q\left(\frac{\eta}{b}\right) \tag{15}$$

Here $\alpha = b/a$.

$$C_{mn}^{pq}(\alpha) = \beta_{mn}^{pq} \int_{-a}^a \int_{-b}^b \int_{-a}^a \int_{-b}^b V(x, y, \xi, \eta) \times \frac{T_m\left(\frac{x}{a}\right) T_n\left(\frac{y}{b}\right) T_p\left(\frac{\xi}{a}\right) T_q\left(\frac{\eta}{b}\right)}{\sqrt{1 - \frac{x^2}{a^2}} \sqrt{1 - \frac{y^2}{b^2}} \sqrt{1 - \frac{\xi^2}{a^2}} \sqrt{1 - \frac{\eta^2}{b^2}}} d\eta d\xi dy dx \tag{16}$$

where $V(x, y, \xi, \eta)$ in formula (16) is given by (11), with $P = 1$.

Coefficients β_{mn}^{pq} were determined by the general Chebyshev's formula for the orthogonal polynomials (Gradshteyn and Ryzhik, 1969).

$$\beta_{00}^{00} = \frac{1}{\pi^4}, \quad \beta_{i0}^{00} = \beta_{0i}^{00} = \beta_{i0}^{i0} = \beta_{00}^{i0} = \beta_{0i}^{0i} = \beta_{0i}^{0k} = \beta_{00}^{ik} = \beta_{i0}^{i0} = \beta_{0k}^{i0} = \beta_{0k}^{i0} = \beta_{k0}^{0i} = \beta_{k0}^{0i} = \beta_{k0}^{0l} = \beta_{0l}^{ik} = \beta_{k0}^{ik} = \frac{8}{\pi^4}, \quad \beta_{mn}^{ik} = \frac{16}{\pi^4}$$

From this, the relationship (14) multiplied by the relationship (15) must be integrated over the area considered in (17). The integral equation of the studied contact problem (Fig. 1) has been previously considered by Aleksandrov et al. (1976) as follows (17):

$$W(x, y) = \int_{-a}^a \int_{-b}^b V(x, y, \xi, \eta) \rho(\xi, \eta) d\eta d\xi \tag{17}$$

By substituting expressions (14) and (15) into (17) and integrating by ξ and η , then the two parts of Eq. (17) could be multiplied by $\frac{T_i(\frac{x}{a}) T_k(\frac{y}{b})}{\sqrt{1 - \frac{x^2}{a^2}} \sqrt{1 - \frac{y^2}{b^2}}}$ $dy dx$ and again integrated over the area of the rectangular plate.

Since the problem is symmetric along the y -axis, all indices related to y and η take only even values. As a result, the relationship between the coefficients A_i and B_{ik} can be represented in a matrix form (18)

$$[C]\{B\} = \frac{\pi E}{(1 - \nu^2)b} [A] \tag{18}$$

where

$$[C] = \pi^4 \begin{bmatrix} C_{00}^{00} & \frac{1}{2} C_{00}^{02} & \frac{1}{2} C_{00}^{04} & \dots & \frac{1}{4} C_{00}^{44} & \dots \\ \frac{1}{2} C_{02}^{00} & \frac{1}{4} C_{02}^{02} & \frac{1}{4} C_{02}^{04} & \dots & \frac{1}{8} C_{02}^{44} & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots \\ \frac{1}{4} C_{12}^{00} & \frac{1}{8} C_{12}^{02} & \frac{1}{8} C_{12}^{04} & \dots & \frac{1}{16} C_{12}^{44} & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots \\ \frac{1}{4} C_{44}^{00} & \frac{1}{8} C_{44}^{02} & \frac{1}{8} C_{44}^{04} & \dots & \frac{1}{16} C_{44}^{44} & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots \end{bmatrix}$$

$$\{B\}^T = [B_{00} \ B_{02} \ \dots \ B_{12} \ \dots \ B_{44} \ \dots]$$

$$[A]^T = \begin{bmatrix} A_0 S_{00}^0 & 0 & 0 & 0 & 0 & \dots \\ 0 & A_1 S_{11}^1 & 0 & 0 & 0 & \dots \\ A_2 S_{20}^2 & A_2 S_{21}^2 & A_2 S_{22}^2 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

where

$$S_{ik}^0 = \int_{-1}^1 \int_{-1}^1 \frac{T_i\left(\frac{x}{a}\right) T_k\left(\frac{y}{b}\right)}{\sqrt{1 - \frac{x^2}{a^2}} \sqrt{1 - \frac{y^2}{b^2}}} dx dy, \quad S_{ik}^1 = \int_{-1}^1 \int_{-1}^1 x \frac{T_i\left(\frac{x}{a}\right) T_k\left(\frac{y}{b}\right)}{\sqrt{1 - \frac{x^2}{a^2}} \sqrt{1 - \frac{y^2}{b^2}}} dx dy, \quad S_{ik}^2 = \int_{-1}^1 \int_{-1}^1 W_2(x, y) \frac{T_i\left(\frac{x}{a}\right) T_k\left(\frac{y}{b}\right)}{\sqrt{1 - \frac{x^2}{a^2}} \sqrt{1 - \frac{y^2}{b^2}}} dx dy.$$

Thus, from (18) the values of the vector $\{B\}$ were obtained.

$$\{B\} = \frac{\pi E}{(1-\nu^2)b} [A] [\Omega] \tag{19}$$

$$\begin{cases} B_{00} = \frac{\pi E}{(1-\nu^2)b} \left(A_0 S_{00}^0 \Omega_{11} + A_1 S_{11}^1 \Omega_{12} + A_2 \sum_{i=1}^{\infty} \Omega_{1i} S_{2i}^2 \right) \\ B_{02} = \frac{\pi E}{(1-\nu^2)b} \left(A_0 S_{00}^0 \Omega_{21} + A_1 S_{11}^1 \Omega_{22} + A_2 \sum_{i=1}^{\infty} \Omega_{2i} S_{2i}^2 \right) \\ B_{04} = \frac{\pi E}{(1-\nu^2)b} \left(A_0 S_{00}^0 \Omega_{31} + A_1 S_{11}^1 \Omega_{32} + A_2 \sum_{i=1}^{\infty} \Omega_{3i} S_{2i}^2 \right) \\ \vdots \end{cases} \tag{20}$$

where $[\Omega] = [C]^{-1}$.

As the determination of the coefficients B_{ik} is not complete, the reaction forces in the contact zone defined by the expression (14) cannot yet be determined. This is the major challenge of the contact problems and relies upon the determination of the unknowns A_i , $i = 0, 1, 2$ in (20).

5.2. Study of the total energy of the system

To determine the coefficients A_0 , A_1 and A_2 (12), the total energy of the system is consider. This includes the quarter-space, along with the plate and the external load. As such, the total energy of the system can be represented as the sum of three terms (Alexandrov and Potapov, 1990), as shown in (21)

$$f\chi 1 = U + A + \Pi \tag{21}$$

where

$$U = \frac{D}{2} \int_{-a}^a \int_{-b}^b \left[\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right)^2 - 2(1-\nu_p) \left(\frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} - \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 \right) \right] dy dx \tag{22}$$

$$A = \frac{1}{2} \int_x \int_y \rho(x, y) W(x, y) dy dx \tag{23}$$

$$\Pi = - \int_x \int_y q(x, y) W(x, y) dy dx \tag{24}$$

U is the energy of bending plate; A the work of reactive forces in the contact zone numerically equal to the energy of deformation of the quarter-space's surface (Selvadura, 1979); Π the work of the external load $q(x, y)$, acting on the plate; D the cylindrical rigidity of the plate; ν_p the Poisson ratio of the plate; $q(x, y)$ is the external load applied on the plate.

By substituting (12), (14) and (20) into (22)–(24) and integrating over the area of the plate, the total energy $f\chi 1$ is obtained. Differentiating the total energy over the unknown coefficients A_i , $i = 0, 1, 2$, a linear system of algebraic equations is obtained for their determination, i.e.

$$\begin{cases} \frac{\partial f\chi 1}{\partial A_0} = 0 \\ \frac{\partial f\chi 1}{\partial A_1} = 0 \\ \frac{\partial f\chi 1}{\partial A_2} = 0 \end{cases} \tag{25}$$

The determination of the unknowns A_i not only allows the determination of deflection of the plate, but also the coefficients B_{ik} , by the formulas (20), which finally allows to determination of the reactive forces in the contact zone.

6. Example calculation

To better demonstrate the efficiency of this approach, a sample calculation is provided. In this case a rectangular plate with cylindrical rigidity D and dimensions $a = b/2 = 1$ m is considered. The plate is located on the surface of an elastic quarter-space with constant E and ν offset at a distance of $oc = 1$ m (Fig. 1). The plate is loaded at its center with a concentrated force P .

Accordingly, the distribution law of the reactive forces (14) takes the form

$$\rho(x, y) = \frac{1}{\sqrt{1-\frac{x^2}{a^2}} \sqrt{1-\frac{y^2}{b^2}}} \sum_{i=0}^2 \sum_{k=0}^1 B_{ik} T_i \left(\frac{x}{a} \right) T_{2k} \left(\frac{y}{b} \right) \tag{26}$$

As a result of the calculations by the proposed Ritz-based method, the following values were obtained:

$$[C] = \begin{bmatrix} 188.366 & -3.604 & 10.156 & 14.559 & .139 & 10.538 \\ -3.604 & 40.297 & -.113 & .139 & 14.24 & .00323 \\ 10.0256 & -.113 & 24.0455 & 10.43 & .00323 & 12.0 \\ 14.354 & .139 & 10.444 & 34.988 & -.1664 & 9.846 \\ .139 & 14.09 & .00325 & -.1664 & 19.88 & -.0145 \\ 10.339 & .00325 & 11.885 & 9.75 & -.0145 & 14.615 \end{bmatrix}$$

$$\{B\}^T = [B_{00} \ B_{10} \ B_{20} \ B_{02} \ B_{12} \ B_{22}]$$

$$[A]^T = \begin{bmatrix} \pi^2 A_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\pi^2}{2} A_1 & 0 & 0 & 0 & 0 \\ -1.53252 A_2 & 0 & -.6502 A_2 & -.6502 A_2 & 0 & 1.5785 A_2 \end{bmatrix}$$

Similarly, by applying (20), the unknown vector $\{B\}$ can be determined:

$$B_{00} = \frac{\pi E}{b(1-\nu^2)} (0.0553162 A_0 + 0.0034509 A_1 - 0.011753 A_2)$$

$$B_{10} = \frac{\pi E}{b(1-\nu^2)} (0.0069 A_0 + 0.163832 A_1 - 0.001643 A_2)$$

$$B_{20} = \frac{\pi E}{b(1-\nu^2)} (-0.00348599 A_0 + 0.00149474 A_1 - 0.1271 A_2)$$

$$B_{02} = \frac{\pi E}{b(1-\nu^2)} (-0.0141455 A_0 - 0.002455 A_1 - 0.0463242 A_2)$$

$$B_{12} = \frac{\pi E}{b(1-\nu^2)} (-0.00541677 A_0 - 0.116164 A_1 + 0.00106289 A_2)$$

$$B_{22} = \frac{\pi E}{b(1-\nu^2)} (-0.0268667 A_0 - 0.00217 A_1 + 0.250593 A_2)$$

From the above, the work of the reaction forces, the energy of bending, and the work of the external load can be determined, as demonstrated below.

6.1. Work of the reaction forces

The work of the reaction forces as expressed in (23) is shown below:

$$\begin{aligned} A &= \frac{1}{2} \int_{-a}^a \int_{-b}^b \left(A_0 + A_1 \frac{x}{a} + A_2 W_2(x, y) \right) \sum_{i=0}^2 \sum_{k=0}^1 B_{ik} T_i \left(\frac{x}{a} \right) T_{2k} \left(\frac{y}{b} \right) \\ &\quad \times \frac{dy dx}{\sqrt{1-\frac{x^2}{a^2}} \sqrt{1-\frac{y^2}{b^2}}} = \frac{E \pi a}{1-\nu^2} \left[0.272975 A_0^2 + 0.404239 A_1^2 \right. \\ &\quad \left. - A_0 (-0.034 A_1 + 0.11586 A_2) - 0.0081 A_1 A_2 + 0.26317 A_2^2 \right] \end{aligned}$$

6.2. The energy of bending

The energy of bending is similarly expressed by use of (22).

$$U = \frac{D}{2} \int_{-a}^a \int_{-b}^b \left[\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right)^2 - 2(1 - \nu_p) \left(\frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} - \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 \right) \right] dy dx$$

$$= \frac{D}{a^3 b^3} \left[(10.549a^4 + 10.549b^4 + a^2 b^2 (35.9 - 14.802\nu_p)) A_2^2 \right]$$

6.3. The work of the external load

Similarly (24) is used to describe the work of the external load

$$\Pi = -PW(x, y) \Big|_{\substack{x=0 \\ y=0}} = -P \left[A_0 + A_2 \left(1 - \frac{[\sin(2.36502)]^2}{[\sinh^2(2.36502)]^2} \right) \right]$$

According to (25)

$$\begin{bmatrix} 0.545949 & 0.034 & -0.11586 \\ 0.034 & 0.808479 & -0.0081 \\ -0.11586 & -0.0081 & 0.526 + \frac{1}{\beta} \left(\frac{21.098a^3}{b^3} + \frac{71.8a}{b} + \frac{21.098b}{a} - \frac{29.6a\nu_p}{b} \right) \end{bmatrix} \times \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0.982349 \end{bmatrix}$$

$\beta = \frac{\pi E a^3}{D(1-\nu^2)}$ is the stiffness ratio (Selvadura, 1979; Gorbunov-Posad et al., 1984).

The solution of this system of equations gives

$$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} = \frac{P(1-\nu^2)}{\pi E a} \times \begin{bmatrix} 1.8365 + \frac{0.0609057ab^3\beta}{5.07073a^4 + 5.07073b^4 + 0.1206ab^3\beta + a^2b^2(17.2568 - 7.1153\nu_p)} \\ -0.07737 + \frac{0.000310355ab^3\beta}{5.07073a^4 + 5.07073b^4 + 0.1206ab^3\beta + a^2b^2(17.2568 - 7.1153\nu_p)} \\ \frac{0.287088ab^3\beta}{5.07073a^4 + 5.07073b^4 + 0.1206ab^3\beta + a^2b^2(17.2568 - 7.1153\nu_p)} \end{bmatrix}$$

This mathematical manipulation in terms of the stiffness ratio, β , easily allows knowing the plate rigidity; the value of β is zero, if the plate is rigid, and it is greater than zero, if the plate is deformable.

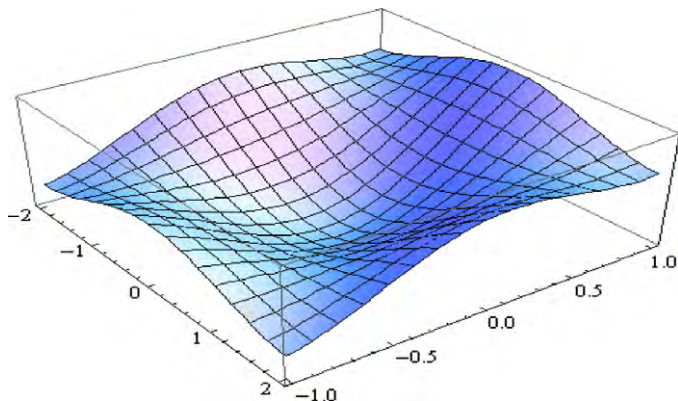


Fig. 7. Vertical displacements depicting the form of a bent rectangular plate resting on the quarter-space's surface due to the application of a vertical concentrated force in the center of the plate.

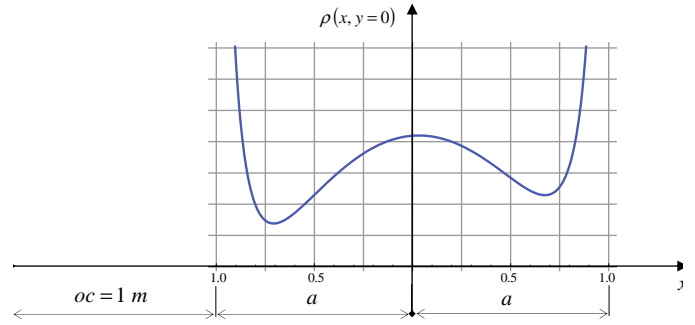


Fig. 8. Reactive force distribution in the contact zone plate and quarter-space's surface at $y = 0$ due to the action of a vertical concentrated force applied at the center of the plate.

Figs. 7 and 8 show the vertical displacements of the bending rectangular plate resting on the surface of an elastic quarter-space depending on $\frac{P(1-\nu^2)}{\pi E a}$ and the reactive force distribution in the contact zone throughout axis x , i.e. at $(y = 0)$ depending on P/a^2 when $\beta = 10$ and $\nu_p = 0.17$.

7. Validation

For a rectangular plate resting on the surface of an elastic quarter-space (Fig. 1), three equations of equilibrium can be created (27):

$$\sum z = \int_{-a}^a \int_{-b}^b \rho(x, y) dx dy = P \tag{27.a}$$

$$\sum M_x = \int_{-a}^a \int_{-b}^b y \rho(x, y) dx dy = 0 \tag{27.b}$$

$$\sum M_y = \int_{-a}^a \int_{-b}^b x \rho(x, y) dx dy = 0 \tag{27.c}$$

By exploiting the orthogonality of the problem (generated by the Chebyshev polynomials) allows direct determination of B_{jk} from formula (26). Thus, for a plate with dimensions $b = 2a$, the following can be obtained from (26)

$$B_{00} = \frac{P}{\pi^2 ab} = 0.05066 \frac{P}{a^2}, \quad B_{10} = 0.$$

The adopted reactive force distribution law in the above example enables the advantageous automatic execution of the second equation of equilibrium (27.b), because of the parity on y .

For the remaining two coefficients in the example, the following values were obtained from the proposed Ritz-based method:

$$B_{00} = 0.05066 \frac{P}{a^2}, \quad B_{10} = 2.08283 \times 10^{-7} \frac{P}{a^2}$$

The exact match of the first two terms confirms the correctness of the calculations. The resulting error (2.08283×10^{-7}) should be explained by the numerical integration errors in the calculation of $C_{mn}^{pq}(\alpha)$ by the formula (16).

Another point of validation is for a rigid, rectangular plate resting on an elastic quarter-space's surface, which can be obtained from the above example at $\beta = 0$:

$$A_0 = 0.58458 \frac{P(1-\nu^2)}{Ea}, \quad A_1 = -0.02463 \frac{P(1-\nu^2)}{Ea}, \quad A_2 = 0$$

For a similar rigid rectangular plate resting on an elastic half-space's surface, as previously reported by Gorbunov-Posad et al. (1984) and Kiselev (1973):

$$w_0 = 0.3182 \frac{P(1-\nu^2)}{Ea}, \quad \varphi_0 = 0.$$

On an elastic quarter-space at $x = y = 0$, the vertical displacement $w_0 = A_0$, and its angle of rotation is relative to the axis $oy \varphi_0 = -A_1/a$. So, linear and rotational displacement of a rigid rectangular plate on an elastic quarter-space is greater than that of the same rigid plate on an elastic half-space. This clearly illustrates that a quarter-space solution should provide a superior result.

8. Conclusions

In this paper, an approach is presented that employs Ritz’s method to calculate the deflections of a deformable rectangular plate resting on an elastic foundation due to a load applied at any point(s) on the plate. This calculation also allows determination of the reactive forces in the contact zone, thereby solving the main challenge of the contact problem. This approach implicitly uses the method of orthogonal polynomials, which allows feature selection of the reactive forces at the edges of the plate. The final algorithms obtained are given in a simple form and compatible with the applications in engineering. It can be realized for any type (model) of elastic foundation, arbitrary external loading and any plate stiffness of a rectangular geometry.

The major advantage of this work lies in the following points:

1. It employs a quarter-space approach, where few researchers have investigated this model due to its complexity; the tendency, generally, is to use the half-space as evidenced by the very high number of publications employing the half-space model
2. It addresses the problem of a deformable plate resting on a foundation, where most research has been restricted to rigid plates.

The proposed solution is almost exact, thereby reducing the maximum error of calculation and in a form sufficiently simple to be compatible with engineering applications.

Appendix A

In the framework of linear theory of bending plates (Kirchhoff theory), the natural oscillations (Eigen-forms) of a rectangular plate ($-a \leq x \leq a$, $-b \leq y \leq b$) with the free edges, cylindrical rigidity mD and Poisson’s ratio ν_p are examined.

The equation of flexural vibrations has the form

$$\Delta \Delta W - \lambda W = 0 \tag{1}$$

where

$$\lambda = m\omega^2/D, \quad \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2,$$

$W(x, y)$ are the plate deflections, m is the mass distribution, ω is the frequency of natural oscillations (Eigen values).

The static boundary conditions are

$$\begin{aligned} x = \pm a : \quad \frac{\partial^2 W}{\partial x^2} + \nu_p \frac{\partial^2 W}{\partial y^2} = \frac{\partial^3 W}{\partial x^3} + (2 - \nu_p) \frac{\partial^3 W}{\partial x \partial y^2} = 0 \\ y = \pm b : \quad \frac{\partial^2 W}{\partial y^2} + \nu_p \frac{\partial^2 W}{\partial x^2} = \frac{\partial^3 W}{\partial y^3} + (2 - \nu_p) \frac{\partial^3 W}{\partial x^2 \partial y} = 0 \end{aligned} \tag{2}$$

Without loss of generality, the case of symmetrical oscillations regarding the axis x and y is considered. Representing the deflections of the plate as the sum of two partial solutions of Eq. (1) as follow:

$$W(x, y) = C_1 \cos(\alpha x) \cos(\beta y) + C_2 ch(\alpha x) ch(\beta y) \tag{3}$$

Substituting (3) in the static boundary conditions (2) and revealing the determinant, (4) is known from the theory of beam functions of transcendental equations for determining α and β .

$$\begin{aligned} \tanh(\alpha) + \tan(\alpha) = 0 \\ \tanh(\beta) + \tan(\beta) = 0 \end{aligned} \tag{4}$$

After substituting the solutions of the system Eq. (4) in the Eq. (1) the expression (5) for the natural frequencies (Eigen values) of vibration symmetrically regarding the axes x and y can be written.

$$\omega_{ik} = \left(\frac{\alpha_i^2}{a^2} + \frac{\beta_k^2}{b^2} \right) \sqrt{\frac{D}{m}} \tag{5}$$

To determine the forms of oscillations (Eigen forms) of the plate, equilibrium conditions equal to zero the torsion moments at the corners of a rectangular plate with free edges are considered (6)

$$x = \pm 1, y = \pm 1 : \quad \frac{\partial^2 W}{\partial x \partial y} = 0 \tag{6}$$

Under condition (6), Eq. (3) leads to the following expression: $\frac{C_2}{C_1} = -\frac{\sin \alpha \sin \beta}{sh \alpha sh \beta}$, that allows defining the forms ik of the symmetrical natural oscillations regarding the coordinate axes x and y as follow:

$$W_{ik}(x, y) = C_1 \left(\cos(\alpha_i x) \cos(\beta_k y) - \frac{\sin(\alpha_i) \sin(\beta_k)}{sh(\alpha_i) sh(\beta_k)} ch(\alpha_i x) ch(\beta_k y) \right) \tag{7}$$

Natural frequencies and forms of natural oscillations corresponding to symmetric and non-symmetric along one or two axes can be obtained similarly. In Table 1, the expressions are given concerning the forms of natural oscillations and the appearance of transcendental equations to determine the natural frequencies according to Eq. (5). It should be borne in mind that the first natural forms of oscillations of a rectangular plate with free edges correspond to the values $\alpha = \beta = 0$, and under such conditions, the displacement and the rotation of the plate appear like those of an absolutely rigid plate.

Table 1
Coordinate functions using in the calculation of deformable rectangular plates.

Fluctuation descriptions	Eigen functions	Transcendental equations and roots
Symmetric regarding axes x and y	$\cos(\alpha_i x) \cos(\beta_k y) - \frac{\sin(\alpha_i) \sin(\beta_k)}{sh(\alpha_i) sh(\beta_k)} ch(\alpha_i x) ch(\beta_k y)$	$\tanh(\alpha) + \tan(\alpha) = 0$ $\tanh(\beta) + \tan(\beta) = 0$ $\alpha_1 = 0 = \beta_1$ $\alpha_2 = 2.3650 = \beta_2$ $\alpha_3 = 5.4978 = \beta_3$...
Symmetric regarding the x -axis	$\cos(\alpha_i x) \sin(\beta_k y) + \frac{\sin(\alpha_i) \cos(\beta_k)}{sh(\alpha_i) ch(\beta_k)} ch(\alpha_i x) sh(\beta_k y)$	$\tanh(\alpha) + \tan(\alpha) = 0$ $\tanh(\beta) - \tan(\beta) = 0$
Symmetric regarding the y -axis	$\sin(\alpha_i x) \cos(\beta_k y) + \frac{\cos(\alpha_i) \sin(\beta_k)}{ch(\alpha_i) sh(\beta_k)} sh(\alpha_i x) ch(\beta_k y)$	$\tanh(\alpha) - \tan(\alpha) = 0$ $\tanh(\beta) + \tan(\beta) = 0$
Non-symmetric regarding axes x and y	$\sin(\alpha_i x) \sin(\beta_k y) - \frac{\cos(\alpha_i) \cos(\beta_k)}{ch(\alpha_i) ch(\beta_k)} sh(\alpha_i x) sh(\beta_k y)$	$\tanh(\alpha) - \tan(\alpha) = 0$ $\tanh(\beta) - \tan(\beta) = 0$

The Eigen functions bring in the above table are used like coordinate functions even for calculation of deformable rectangular plates resting on elastic foundation as our case, whatever the case of symmetry or non-symmetry regarding the axes of co-ordinates.

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