MINISTRY OF EDUCATION OF THE REPUBLIC OF BELARUS
Belarusian National Technical University

Department “Engineering mathematics”

M. A. Hundzina,
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MATHEMATICS.
MATRICES AND VECTORS

Educational-methodical manual

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Educational-methodical manual
for students of specialities
6-05-0716-03 “Information and measuring instruments and systems”,
6-05-0716-04 “Optoelectronic and laser technology”,
6-05-0716-06 “Biomedical engineering”

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The educational-methodical manual is intended for students of the first stage of training of the instrument-building faculty of BNTU, studying the discipline “Mathematics”.

The manual contains materials for organizing a system of continuous development in the sections “Basic operations with matrices”, “Vector algebra”.

The given set of tasks for lectures, practical work and consultations was developed taking into account the recommendations of the Department of Engineering Mathematics of the Instrument-Making Faculty of the Belarusian National Technical University and is consistent with the requirements for the level of training of specialists.

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INTRODUCTION

The educational-methodical manual “Mathematics. Matrices and vectors” is intended for lectures, practical exercises and consultations with students of the first stage of training in the specialities 6-05-0716-03 “Information and measuring instruments and systems”, 6-05-0716-04 “Optoelectronic and laser technology”, 6-05-0716-06 “Biomedical engineering” of the instrument-making faculty of the Belarusian National Technical University in the discipline “Mathematics”.

This training manual includes the main topics necessary for the formation of the appropriate competence of a specialist, from the sections “Operations on matrices”, “Vector algebra”. These competencies must be mastered by the student during the academic semester for further successful mastering of the material in related special disciplines.

The topics covered by this manual correspond to the current curriculum for the discipline “Mathematics” for the instrument-making faculty of the Belarusian National Technical University.

The authors of the teaching aid aimed at increasing the level of mastering the educational material, the student's independence in preparation for the exam in this discipline, the implementation of the basic principles of didactics: the accessibility and consistency of the educational process. Careful selection of material allows for the primary consolidation of the material, as well as systematize the knowledge of students.

§ 1 MATRICES AND MATRIX OPERATIONS. DIAGONAL, TRIANGULAR AND SYMMETRIC MATRICES

A matrix is defined to be a rectangular array of functional or numeric elements, arranged in row or column order. Most important in this definition is that two subscripts are required to identify a given element: a row subscript and a column subscript (fig.1).

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

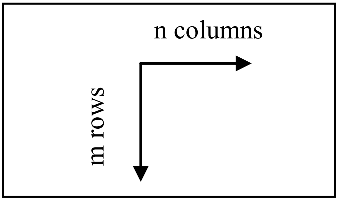


Fig. 1. Matrix A:
(m, n) – dimension of matrix

A square matrix is a matrix with the same number of rows and columns.

You learned numbers at the school. Even at the beginning of the education, you learned how to add numbers, how to multiply them, find the reverse number.

We have a new object today – a matrix. We don't know anything about it. We have to learn how to use matrices. We will learn how to add, multiply. We will answer the question whether it is possible to divide matrices, how to find their inverse.

Where you can meet the matrix? It is everywhere! In front of you is a phone or computer screen. It’s no secret for you that the screen consists of many pixels. Before our eyes is a matrix of pixels where the pixel brightness value is located at the intersection of rows and columns.

You add effects to photos often in phone applications. It means that you interact with image matrices. You press a button and the computer does all the work for you. A real engineer must know what happens when he pushes a button on a device developed by him.

We will see how you can interact with matrices and use them, for example, to solve systems of equations.

Example.

The matrix can contain an unequal number of rows and columns. It can have only one row, and it can only have one column. Here are examples of matrices.

$$A = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \; B = \begin{pmatrix} \sqrt{2} & 6 \end{pmatrix}, \; C = \begin{pmatrix} 8 & -1 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}.$$

Example.

Write out the elements of a given matrix a_{11}, a_{32}, a_{21}.

$$A = \begin{pmatrix} 8 & 1 \\ 10 & 2 \\ 1 & 3 \end{pmatrix}.$$

The main diagonal of a matrix consists of those elements that lie on the diagonal that runs from top left to bottom right. The main diagonal starts at the top left and goes down to the right:

$$\begin{pmatrix} 1 & 7 & 5 \\ 8 & 2 & 0 \\ -9 & 1 & 10 \end{pmatrix} \begin{pmatrix} 8 & 1 \\ 10 & 2 \\ 1 & 3 \end{pmatrix}.$$

A diagonal matrix is a matrix, in which the entries outside the main diagonal are all zero; the term usually refers to square matrices.

A square matrix is called lower triangular if all the entries above the main diagonal are zero. Similarly, a square matrix is called upper triangular if all the entries below the main diagonal are zero.

§ 2 ADDITION OF THE MATRICES

The sum of two (or more) matrices is formed by summing corresponding elements:

$$C = A \pm B,\ c_{ij} = a_{ij} \pm b_{ij}.$$

Example.

Find the sum $A + B$, the difference of the matrices $A - B$.

$$A = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 6 & -2 \\ 5 & -4 & 1 \end{pmatrix},\ B = \begin{pmatrix} 10 & 2 & 3 \\ 2 & 0 & -1 \\ 1 & -2 & -1 \end{pmatrix}.$$

Matrix addition is defined only when A and B have the same numbers of rows and columns, respectively.

When this is the case, the matrices A and B are said to be "conformable in addition".

If all the elements of A are respectively the negatives of those of B, then the sum, C, will have all zero elements. In such case, C is known as a "nul" matrix.

A square matrix whose ij elements are zero for $i \neq j$ and whose elements ii are unity is defined as the "unit matrix".

Since addition is commutative for the elements of the matrix, then matrix addition itself is commutative. That is,

$$A + B = B + A.$$

§ 3 MULTIPLICATION BY A SCALAR

The matrix $k{\cdot}A$ is formed by multiplying every element of A by the scalar k.

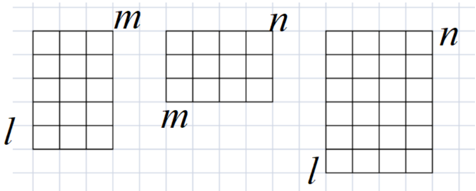
Example.

Evaluate expression $7A + 3B$.

$$A = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 9 & -4 & 1 \end{pmatrix},\ B = \begin{pmatrix} 10 & 2 & 3 \\ 2 & 0 & -1 \\ 1 & -2 & -1 \end{pmatrix}.$$

§ 4 MATRIX MULTIPLICATION

For matrix multiplication, the number of columns in the first matrix must be equal to the number of rows in the second matrix. The resulting matrix, known as the matrix product, has the number of rows of the first and the number of columns of the second matrix (fig. 2).



$A \cdot B = C.$

Fig. 2. Type of matrix

That is, the elements of C as the product are obtained by multiplying term-by-term the elements of the ith row of A and the jth column of B, and summing these n products.

In other words, C is the dot product of the ith row of A and the jth column of B.

We match the 1st members (1 and 7), multiply them, likewise for the 2nd members (2 and 9) and the 3rd members (3 and 11), and finally sum them up.

Dot product

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 58 & * \\ * & * \end{pmatrix}$$

We see on the first row of A and second column of B. We match the 1st members (1 and 8), multiply them, likewise for the 2nd members (2 and 10) and the 3rd members (3 and 12), and finally sum them up.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 58 & 64 \\ * & * \end{pmatrix}$$

We can do the same operations for the 2nd row and 1st column.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 58 & 64 \\ 134 & 154 \end{pmatrix}$$

Example.
Find the product of matrices, if possible.

a) $A = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 1 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ -2 & 3 \\ 4 & 2 \end{pmatrix}.$

b) $A = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 5 & -2 \\ 5 & -4 & 1 \end{pmatrix}, B = \begin{pmatrix} 10 & 2 & 3 \\ 2 & 10 & -1 \\ 1 & -2 & -1 \end{pmatrix}.$

c) $A = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, B = \begin{pmatrix} 7 & 5 & -2 \end{pmatrix}.$

Example.
Find the product of matrices $A \cdot B$, $B \cdot A$, if possible.

$$A = \begin{pmatrix} -2 & 3 \\ 5 & 4 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}.$$

The results are different, which illustrates that matrix multiplication is not commutative. That is, in general:

$$A \cdot B \neq B \cdot A.$$

Because of the non-commutative nature of the matrix product, the order of the product must be stated explicitly.

Matrix multiplication is associative:

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C = A \cdot B \cdot C.$$

Further, it is distributive:

$$A \cdot (B + C) = A \cdot B + A \cdot C.$$

§ 5 MATRIX TRANSPOSITION

We can swap elements across the main diagonal (rows become columns).

The matrix transpose of A is written A^{T}. A^{T} is obtained by interchanging the rows and columns of A. If A is an $m \times n$ *(m by n)* matrix, then A^{T} is an $n \times m$ matrix.

A square matrix whose transpose is equal to itself is called a symmetric matrix.

Properties.

Let A and B be matrices and c be a scalar.

1. The operation of taking the transpose is an involution (self-inverse).

$$(A^{\mathrm{T}})^{\mathrm{T}} = A.$$

2. The transpose respects addition.

$$(A + B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}.$$

3. Note that the order of the factors reverses.

$$(A \cdot B)^{\mathrm{T}} = B^{\mathrm{T}} \cdot A^{\mathrm{T}}.$$

4. The number can be placed outside the transposition sign.

$$(c \cdot A)^{\mathrm{T}} = c \cdot A^{\mathrm{T}}.$$

Example.
Find transposed matrix A^{T}.

a) $A = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 5 & -2 \\ 5 & -4 & 1 \end{pmatrix}$, b) $B = \begin{pmatrix} 10 & 2 & 3 \\ 2 & 10 & -1 \\ 1 & -2 & -1 \end{pmatrix}$.

Example.

Find $2A^2 - 3A^{\mathrm{T}}$, where $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 9 & -2 \\ 5 & 4 & 1 \end{pmatrix}$.

TASK TO REVIEW

1. Find the product of two matrices $A{\cdot}B$ and $B{\cdot}A$, if possible.

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 5 & 0 & -1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}.$$

2. Transpose matrices

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -2 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}.$$

3. Find the value of the expression $A^2 + 7 \cdot B$.

$$A = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -2 & 2 \\ 2 & -1 & 0 \end{pmatrix}.$$

4. Find the value of the expression $A^{\mathrm{T}}\cdot B-5\cdot E$, where E – unit matrix.

$$A=\begin{pmatrix}1 & 3\\ -2 & -1\\ 3 & 1\end{pmatrix},\ B=\begin{pmatrix}-5 & 3\\ -1 & -4\\ 3 & 1\end{pmatrix}.$$

5. Find the value of the expression $(C\cdot D+D^{\mathrm{T}}\cdot C^{\mathrm{T}})^3$, where E – unit matrix.

$$C=\begin{pmatrix}-1 & -2\\ -1 & -3\end{pmatrix},\ D=\begin{pmatrix}-1 & 1\\ 1 & 2\end{pmatrix}.$$

§ 6 DETERMINANTS

The determinant is a scalar value that can be computed from the elements of a square matrix and reflects certain properties of the linear transformation described by the matrix.

The determinant helps us find the inverse of a matrix, tells us things about the matrix that are useful in systems of linear equations, calculus.

The symbol for determinant is two vertical lines on either side.

$$|A|=\begin{vmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \dots & \dots & \dots & \dots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{vmatrix}.$$

The value of a second-order determinant is equal to the product of the elements on the main diagonal, minus the product of the elements on the secondary diagonal. The formula for finding the determinant of the second order:

$$\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc\ .$$

Example.

$$\begin{vmatrix} 7 & 4 \\ 3 & 1 \end{vmatrix} = 7\cdot 1 - 3\cdot 4 = 7 - 12 = -5\,.$$

Sarrus' rule is useful for third-order determinants only (fig. 3). Once this is done the calculation of the determinant is computed as follows: multiply the diagonal elements. The descending diagonal from left to right has a “+” sign, while the descending diagonal from right to left has the “–” sign.

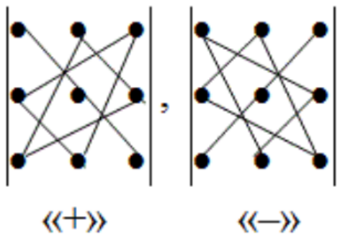


Fig. 3. Sarrus' rule

Example.

$$\begin{vmatrix} 4 & -1 & 0 \\ 2 & 1 & -1 \\ 6 & 2 & 1 \end{vmatrix} = 4\cdot 1\cdot 1 + 2\cdot 2\cdot 0 + 6\cdot(-1)\cdot(-1) - 0\cdot 1\cdot 6 - 4\cdot 2\cdot(-1) -$$

$$-2\cdot 1\cdot(-1) = 4 + 0 + 6 - 0 + 8 + 2 = 20.$$

§ 7 PROPERTIES OF DETERMINANTS

1. A square matrix A and its transpose A^{T} have the same determinant.

2. If any row, or column, of a determinant contains all zero elements, that determinant equals zero:

$$|A| = 0.$$

3. The determinant of a diagonal matrix is equal to the product of its diagonal elements.

4. If two rows, or columns, of a determinant are interchanged, the sign of the determinant is reversed.

5. If two rows, or columns, of a determinant are identical, its expansion is zero.

6. If to any row, or column, there is added a constant factor multiplied by the corresponding elements of any other row, or column, the value of the determinant is unchanged.

Example.

Calculate the determinant of this matrix:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & 2 \\ 0 & -3 & 1 \end{pmatrix}.$$

§ 8 MINORS AND COFACTORS

If one, or more, rows and columns are deleted from a determinant, the result is a determinant of lower order and is called a “minor” of the original. If just one row and one column are deleted, the resulting “first minor” is of order $(n - 1)$.

Minors obtained by removing just one row and one column from square matrices (first minors) are required for calculating matrix cofactors, which in turn are useful for computing both the determinant and inverse matrix.

Example.

Calculate the minors M_{11}, M_{23}, M_{31} of this matrix:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & 2 \\ 0 & -3 & 1 \end{pmatrix}.$$

The cofactor A_{ij} of a_{ij} is defined by

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

One of the main applications of cofactors is finding the determinant. The following theorem, which we will not prove, shows us how to use cofactors to find a determinant.

Theorem.

Let A be an $n \times n$ matrix and $1 \le i \le n$. Then

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}.$$

Example.

Calculate the determinant of matrix A, using the expansion in terms of the elements of the first row:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & 2 \\ 0 & -3 & 1 \end{pmatrix}.$$

The determinant will be equal to the sum of the products of elements by their cofactors.

TASK TO REVIEW

1. Calculate the determinant $\begin{vmatrix} \cos x & \sin x \\ \sin x & -\cos x \end{vmatrix}$.

2. Calculate the minors M_{11}, M_{23}, M_{31} of this matrix:

$$A = \begin{pmatrix} -2 & 3 & 5 \\ 7 & -1 & 4 \\ 9 & -8 & -6 \end{pmatrix}.$$

3. Calculate the cofactors A_{11}, A_{23}, A_{31} of this matrix:

$$A = \begin{pmatrix} -1 & 0 & 2 \\ 3 & 1 & 1 \\ 2 & -3 & 4 \end{pmatrix}.$$

4. Calculate the determinant of matrix A, using the expansion in terms of the elements of the first row:

$$A = \begin{pmatrix} -2 & 3 & 5 \\ 7 & -1 & 4 \\ 9 & -8 & -6 \end{pmatrix}.$$

5. Calculate the determinant of matrix A, using Sarrus' rule:

$$A = \begin{pmatrix} -2 & 3 & 5 \\ 7 & -1 & 4 \\ 9 & -8 & -6 \end{pmatrix}.$$

§ 9 INVERSE MATRIX

Thus far, we have not defined matrix division. In the general case, no such operation as A/B exists (fig. 4).

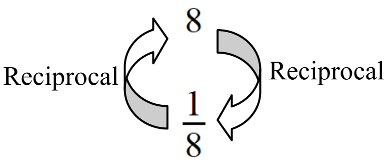


Fig. 4. Reciprocal number

However, if A is a square matrix, then there may be a matrix, B, such that $A \cdot B = E$, E – unit matrix. In this case, the matrix B is referred to as the “inverse” of A, and is written with A^{-1} in superscript (fig. 5).

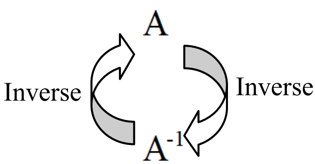


Fig. 5. Inverse matrix

The notation A/B or $A = 1/B$ is never used. The matrices, A and B, shown below, are examples:

$$A=\begin{pmatrix}1 & 2 & 2\\ 1 & 0 & 1\\ 1 & -3 & 0\end{pmatrix}, B=\begin{pmatrix}-3 & 6 & -2\\ -1 & 2 & -1\\ 3 & -5 & 2\end{pmatrix}, A\cdot B=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

$$B = A^{-1}.\ A \cdot A^{-1} = A^{-1} \cdot A = E.$$

The inverse of A is given by

$$A^{-1} = \frac{1}{|A|} A^{*},$$

where A^* is a matrix composed of cofactors written in columns.

The necessary and sufficient condition for the existence of the inverse of a square matrix A is that $|A| \neq 0$.

Example.

1. Calculate the inverse matrix for

$$A=\begin{pmatrix}-2 & 3 & 5\\ 7 & -1 & 4\\ 9 & -8 & -6\end{pmatrix}.$$

2. Find inverse matrix for

$$A=\begin{pmatrix}1 & -1 & 1\\ 2 & 1 & 1\\ 1 & 1 & 2\end{pmatrix}.$$

3. Find the product

$$\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}\cdot\begin{pmatrix}3 & 6 & 4\\ 2 & 3 & -5\\ 0 & -8 & 1\end{pmatrix}.$$

§ 10 CRAMER'S RULE

Given a system of linear equations, Cramer's Rule is a handy way to solve system of equations.

Let's use the following system of equations:

$$\begin{aligned} 2x + y + z &= 3, \\ x - y - z &= 0, \\ x + 2y + z &= 0. \end{aligned}$$

We have the left-hand side of the system with the variables (the "coefficient matrix") and the right-hand side with the answer values.

Let D be the determinant of the coefficient matrix of the above system, and let Dx be the determinant formed by replacing the x-column values with the answer-column values. Dy is obtained by replacing the second column with the answer column.

System of equations	Coefficient matrix's determinant	Answer column
$2x + 1y + 1z = 3,$ $1x - 1y - 1z = 0,$ $1x + 2y + 1z = 0.$	$D = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$	$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$

Dx: coefficient determinant with answer-column values in x-column	Dy: coefficient determinant with answer-column values in y-column	Dz: coefficient determinant with answer-column values in z-column
$Dx = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix}$	$Dy = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix}$	$Dz = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix}$

Cramer's Rule says that $x = Dx \,/\, D$, $y = Dy \,/\, D$, and $z = Dz \,/\, D$. The rule is used only in the case when the determinant of the system matrix is nonzero.

TASK TO REVIEW

Solve the system of equation by Cramer's Rule:

1. $\begin{cases} 2x_1 + x_2 + 3x_3 = 7, \\ 2x_1 + 3x_2 + x_3 = 1, \\ 3x_1 + 2x_2 + x_3 = 6. \end{cases}$

2. $\begin{cases} 3x_1 - 2x_2 + 4x_3 = 12, \\ 3x_1 + 4x_2 - x_3 = 1, \\ 2x_1 - x_2 - x_3 = 4. \end{cases}$

3. $\begin{cases} 4x_1 + x_2 - 3x_3 = 9, \\ x_1 + x_2 - x_3 = -2, \\ 8x_1 + 3x_2 - 6x_3 = 12. \end{cases}$

4. $\begin{cases} 2x_1 + 3x_2 + 4x_3 = 12, \\ 7x_1 - 5x_2 + x_3 = -33, \\ 4x_1 + x_3 = -7. \end{cases}$

5. $\begin{cases} 2x_1 - x_2 + 2x_3 = 3, \\ x_1 + 2x_2 + 2x_3 = -4, \\ 4x_1 + x_2 + 4x_3 = -3. \end{cases}$

6. $\begin{cases} 3x_1 - 2x_2 - 4x_3 = 21, \\ 3x_1 + 4x_2 - 2x_3 = 9, \\ 2x_1 - x_2 - x_3 = 10. \end{cases}$

7. $\begin{cases} 4x_1 + x_2 + 4x_3 = 19, \\ 2x_1 - 2x_2 + 2x_3 = 11, \\ x_1 + x_2 + 2x_3 = 8. \end{cases}$

8. $\begin{cases} 2x_1 + x_2 + 3x_3 = 7, \\ 2x_1 + 3x_2 + x_3 = 1, \\ 6x_1 + 4x_2 + 2x_3 = 12. \end{cases}$

9. $\begin{cases} 6x_1 - 4x_2 + 8x_3 = 24, \\ 3x_1 + 4x_2 - x_3 = 1, \\ 2x_1 - x_2 - x_3 = 4. \end{cases}$

10. $\begin{cases} 4x_1 + x_2 - 3x_3 = 9, \\ 2x_1 + 2x_2 - 2x_3 = -4, \\ 8x_1 + 3x_2 - 6x_3 = 12. \end{cases}$

§ 11 RANK AND THE FUNDAMENTAL MATRIX SPACES. ELEMENTARY MATRIX OPERATIONS

Elementary matrix operations play an important role in many matrix algebra applications, such as finding the inverse of a matrix and solving systems of equations.

There are three kinds of elementary matrix operations:

1. Interchange two rows (or columns).
2. Multiply each element in a row (or column) by a non-zero number.
3. Multiply a row (or column) by a non-zero number and add the result to another row (or column).

A common approach to finding the rank of a matrix is to reduce it to a simpler form by elementary matrix operations. Row operations do not change the row space. The rank equals the number of non-zero rows in the matrix after application of elementary matrix operations.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{pmatrix} \xrightarrow{2R_1 + R_2 \to R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{pmatrix} \longrightarrow$$

$$\xrightarrow{-3R_1 + R_3 \to R_3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{pmatrix} \xrightarrow{R_2 + R_3 \to R_3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

The final matrix has two non-zero rows and thus the rank of matrix A is 2.

Example.

Find the rank of a matrix:

$$A=\begin{pmatrix} 2 & 5 & 6 \\ 4 & -1 & 5 \\ 2 & -6 & -1 \end{pmatrix}.$$

Example.

Find the rank of a matrix:

$$A=\begin{pmatrix} -2 & 0 & 8 & 1 & -5 \\ 3 & -1 & 7 & 2 & 4 \\ -8 & 2 & -6 & -3 & -13 \\ 11 & -3 & 13 & 5 & 17 \end{pmatrix}.$$

TASK TO REVIEW

Find the rank of the matrix of the system:

$$1.\ \begin{cases} 5x_1-6x_2+4x_3=0, \\ 3x_1-3x_2+x_3=0, \\ 2x_1-3x_2+3x_3=0. \end{cases}$$

$$2.\ \begin{cases} x_1+x_2+x_3=0, \\ 2x_1-3x_2+4x_3=0, \\ 3x_1-2x_2+5x_3=0. \end{cases}$$

$$3.\ \begin{cases} 3x_1-2x_2+3x_3=0, \\ 2x_1+3x_2-4x_3=0, \\ 5x_1+x_2-x_3=0. \end{cases}$$

4. $\begin{cases} x_1 - x_2 + 2x_3 = 0, \\ 2x_1 + x_2 - 3x_3 = 0, \\ 3x_1 - x_3 = 0. \end{cases}$

5. $\begin{cases} 5x_1 - 5x_2 + 4x_3 = 0, \\ 3x_1 + x_2 + 3x_3 = 0, \\ 2x_1 - 6x_2 + x_3 = 0. \end{cases}$

6. $\begin{cases} 2x_1 + x_2 + 3x_3 = 0, \\ x_1 - x_2 + 2x_3 = 0, \\ 3x_1 + 5x_3 = 0. \end{cases}$

7. $\begin{cases} 2x_1 + x_2 - x_3 = 0, \\ 3x_1 - 2x_2 + 4x_3 = 0, \\ 5x_1 - x_2 + 3x_3 = 0. \end{cases}$

8. $\begin{cases} x_1 + 4x_2 - 3x_3 = 0, \\ 2x_1 + 5x_2 + x_3 = 0, \\ x_1 + x_2 + 4x_3 = 0. \end{cases}$

9. $\begin{cases} x_1 + 2x_2 + 3x_3 = 0, \\ 2x_1 - x_2 - x_3 = 0, \\ 3x_1 + x_2 + 2x_3 = 0. \end{cases}$

10. $\begin{cases} 2x_1 - x_2 + 3x_3 = 0, \\ x_1 + 2x_2 - 5x_3 = 0, \\ 3x_1 + x_2 - 2x_3 = 0. \end{cases}$

§ 12 GAUSSIAN ELIMINATION

Gaussian elimination is probably the best method for solving systems of equations if you don’t have a graphing calculator or computer program to help you.

This technique is also called row reduction and it consists of two stages: forward elimination and back substitution.

The goals of Gaussian elimination are to make the upper-left corner element a one, use elementary row operations to get zeros in all positions under first one, get ones for leading coefficients in every row diagonally from the upper-left to lower-right corner, and get zeros under ones. Basically, you eliminate all variables in the last row except for one (the first stage: forward elimination).

Then you can use back substitution to solve for one variable at a time by plugging the values you know into the equations from the bottom up (the second stage: back substitution).

You accomplish this elimination by eliminating the x (or whatever variable comes first) in all equations except for the first one. Then eliminate the second variable in all equations except for the first two. This process continues, eliminating one more variable per row, until only one variable is left in the last row. Then solve for that variable.

Example.

If we were to have the following system of linear equations containing three equations for three unknowns:

$$\begin{cases} x + y + z = 3, \\ x + 2y + 3z = 0, \\ x + 3y + 2z = 3. \end{cases}$$

we can represent such system as an augmented matrix like the one below:

$$\begin{cases} x + y + z = 3, \\ x + 2y + 3z = 0, \\ x + 3y + 2z = 3. \end{cases} \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 3 \end{array}\right)$$

The first stage: forward elimination

Let us row-reduce (use Gaussian elimination) so we can simplify the matrix.

The ith row, multiplied by a number, is added to the jth row.

The ith row (does not change it), multiplied by a number, is added to the jth row (we change it).

$$\left(\begin{array}{ccc|c}1&1&1&3\\1&2&3&0\\1&3&2&3\end{array}\right)\xrightarrow{R_2-R_1\to R_2}\left(\begin{array}{ccc|c}1&1&1&3\\0&1&2&-3\\1&3&2&3\end{array}\right)\xrightarrow{R_3-R_1\to R_3}$$

$$\left(\begin{array}{ccc|c}1&1&1&3\\0&1&2&-3\\0&2&1&0\end{array}\right)\xrightarrow{2R_2\to R_2}\left(\begin{array}{ccc|c}1&1&1&3\\0&2&4&-6\\0&2&1&0\end{array}\right)\xrightarrow{R_3-R_2\to R_3}$$

$$\left(\begin{array}{ccc|c}1&1&1&3\\0&2&4&-6\\0&0&-3&6\end{array}\right).$$

The second stage: back substitution.

$$\left(\begin{array}{ccc|c}1&1&1&3\\0&2&4&-6\\0&0&-3&6\end{array}\right)\longrightarrow\begin{cases}x+y+z=3,\\2y+4z=-6,\\-3z=6.\end{cases}$$

From the last equation we express z. Then we substitute it into the second equation and find y.

$$\begin{cases}x+y+z=3\\2y+4z=-6\\z=-2.\end{cases};\begin{cases}x+y+z=3\\2y-8=-6\\z=-2.\end{cases};\begin{cases}x+y+z=3\\y=1\\z=-2.\end{cases};\begin{cases}x+1-2=3\\y=1\\z=-2.\end{cases};\begin{cases}x=4\\y=1\\z=-2.\end{cases}$$

Answer: (4, 1, 2).

Example.
Solve the system of equation by Gauss reduction method:

$$\begin{cases}7x_1+4x_2-3x_3=13,\\6x_1+4x_2+6x_3=6,\\2x_1-3x_2+3x_3=-10.\end{cases}$$

TASK TO REVIEW

Solve the system of equation by Gauss reduction method:

1. $\begin{cases} 4x_1 - x_2 = -6, \\ x_1 + 2x_2 + 5x_3 = -14, \\ x_1 - 3x_2 + 4x_3 = -19. \end{cases}$

2. $\begin{cases} x_1 + 4x_2 - x_3 = -9, \\ 4x_1 - x_2 + 5x_3 = -2, \\ 3x_2 - 7x_3 = -6. \end{cases}$

3. $\begin{cases} 2x_1 + x_2 + 3x_3 = 7, \\ 2x_1 + 3x_2 + x_3 = 1, \\ 3x_1 + 2x_2 + x_3 = 6. \end{cases}$

4. $\begin{cases} 3x_1 - 2x_2 + 4x_3 = 12, \\ 3x_1 + 4x_2 - x_3 = 1, \\ 2x_1 - x_2 - x_3 = 4. \end{cases}$

5. $\begin{cases} 4x_1 + x_2 - 3x_3 = 9, \\ x_1 + x_2 - x_3 = -2, \\ 8x_1 + 3x_2 - 6x_3 = 12. \end{cases}$

6. $\begin{cases} 2x_1 + 3x_2 + 4x_3 = 12, \\ 7x_1 - 5x_2 + x_3 = -33, \\ 4x_1 + x_3 = -7. \end{cases}$

7. $\begin{cases} 2x_1 - x_2 + 2x_3 = 3, \\ x_1 + 2x_2 + 2x_3 = -4, \\ 4x_1 + x_2 + 4x_3 = -3. \end{cases}$

8. $\begin{cases} 3x_1 - 2x_2 - 4x_3 = 21, \\ 3x_1 + 4x_2 - 2x_3 = 9, \\ 2x_1 - x_2 - x_3 = 10. \end{cases}$

$$9.\ \begin{cases} 4x_1 + x_2 + 4x_3 = 19, \\ 2x_1 - 2x_2 + 2x_3 = 11, \\ x_1 + x_2 + 2x_3 = 8. \end{cases}$$

$$10.\ \begin{cases} 2x_1 - x_2 + 2x_3 = 8, \\ x_1 + x_2 + 2x_3 = 11, \\ 4x_1 + x_2 + 4x_3 = 22. \end{cases}$$

§ 13 VECTOR IN 2-SPACE AND 3-SPACE

A vector is an object that has both a magnitude (length) and a direction. Geometrically, we can picture a vector as a directed line segment, whose length is the magnitude of the vector, with an arrow indicating the direction (fig. 6). The direction is from its tail to its head.

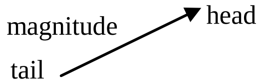


Fig. 6. Vector

Two vectors are the same if they have the same magnitude and direction. This means that if we take $\vec{a}$ a vector and translate it to a new position (without rotating it), then the vector we obtain at the end of this process is the same vector we had in the beginning.

People will sometimes denote vectors using arrows as $\vec{a}$, or they use other markings.

We denote the magnitude of the vector *a* by $|\vec{a}|$ (fig. 7).

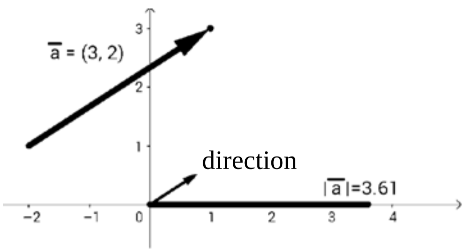


Fig. 7. Vector $\vec{a}$

The bold arrow represents a vector $\vec{a}$. The two defining properties of a vector, magnitude and direction, are illustrated by a bar and a arrow, respectively.

The one exception is when $\vec{a}$ is the zero vector (the only vector with zero magnitude), for which the direction is not defined.

A vector whose magnitude $|\vec{a}|$ is unity is called a "unit vector".

§ 14 OPERATIONS ON VECTORS

We can define a number of operations on vectors geometrically without reference to any coordinate system.

Here we define addition, subtraction, and multiplication by a scalar.

Addition of vectors.

Given two vectors $\vec{a}$ and $\vec{b}$, we $\vec{b}$ is the vector $\vec{a}+\vec{b}$.

Addition of vectors satisfies two important properties:

1. The commutative law, which states the order of addition doesn't matter:

$$\vec{a}+\vec{b}=\vec{b}+\vec{a}.$$

This law is also called the para $\vec{b}+\vec{a}$. But both sums are equal to the same diagonal of the parallelogram.

2. The associative law, which states that the sum of three vectors does not depend on which pair of vectors is added first:

$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c}).$$

Vector subtraction.

Before we define s $\vec{b}$ ubtraction, we define the vector $-\vec{a}$, which is the opposite of $\vec{a}$. The vector $-\vec{a}$ is the vector with the same magnitude as $\vec{a}$ but it is pointed in the opposite direction.

We define subtraction as addition with the opposite of a vector:

$$\vec{b}-\vec{a}=\vec{b}+(-\vec{a}).$$

This is equivalent to turning vector $\vec{a}$ around in the applying the above rules for addition.

Scalar multiplication.

Given a vector $\vec{a}$ and a real number (scalar) k, we can form the vector $k\vec{a}$ as follows. If k is positive, then $k\vec{a}$ is the vector whose direction is the same as the direction of $\vec{a}$ and whose magnitude is k times the magnitude of $\vec{a}$. In this case, multiplication by k simply stretches (if $k > 1$) or compresses (if $k < 1$) the vector $\vec{a}$.

If, on the other hand, k is negative, then we have to take the opposite of $\vec{a}$ before stretching or compressing it.

In other words, the vector $k\vec{a}$ points in the opposite direction of $\vec{a}$, and the magnitude of $k\vec{a}$ is k times the length of $\vec{a}$. No matter the sign of k, we observe that the magnitude of $k\vec{a}$ is k times the magnitude of $\vec{a}$: $|k\vec{a}|=|k||\vec{a}|$.

Scalar multiplication satisfies many of the same properties as the usual multiplication.

1. $s \cdot (\vec{a}+\vec{b}) = s \cdot \vec{a} + s \cdot \vec{b}$ (distributive law).
2. $(s + t) \cdot \vec{a} = s \cdot \vec{a} + t \cdot \vec{a}$ (distributive law).
3. $1 \cdot \vec{a} = \vec{a}$.
4. $(-1) \cdot \vec{a} = -\vec{a}$.
5. $0 \cdot \vec{a} = \vec{0}$.

In the last formula, the zero on the left is the number 0, while the zero on the right is the vector $\vec{0}$, which is the unique vector whose magnitude is zero.

If $\vec{a} = k\vec{b}$ for some scalar k, then we say that the vectors $\vec{a}$ and $\vec{b}$ are parallel. If k is negative, some people say that $\vec{a}$ and $\vec{b}$ are anti-parallel, but we will not use that language.

We are able to describe vectors, vector addition, vector subtraction, and scalar multiplication without reference to any coordinate system.

However, sometimes it is useful to express vectors in terms of coordinates, as discussed about vectors in the standard Cartesian coordinate systems in the plane and in three-dimensional space.

Often a coordinate system is helpful because it can be easier to manipulate the coordinates of a vector rather than manipulating its magnitude and direction directly. When we express a vector in a coordinate system, we identify a vector with a list of numbers, called coordinates or components, that specify the geometry of the vector in terms of the coordinate system.

§ 15 VECTORS IN THE PLANE

We assume that you are familiar with the standard (x, y) Cartesian coordinate system in the plane. Each point p in the plane is identified with its x and y components:

$$p = (p_1, p_2).$$

To determine the coordinates of a vector $\vec{a}$ in the plane, the first step is to translate the vector so that its tail is at the origin of the coordinate system. Then the head of the vector will be at some point (a_1, a_2) in the plane. We call (a_1, a_2) the coordinates or the components of the vector $\vec{a}$. We often write $\vec{a} \in R^2$, to denote that it can be described by two real coordinates (fig. 8).

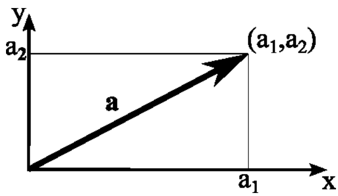


Fig. 8. Vector $\vec{a}$

Using the Pythagorean Theorem, we can obtain an expression for the magnitude of a vector in terms of its components. Given a vector $\vec{a}$ = (a_1, a_2), the vector is the hypotenuse of a right triangle whose legs are length a_1 and a_2. Hence, the magnitude of the vector $\vec{v}$: $\vec{v} = (v_x, v_y, v_z)$ in three dimensions, $\vec{v} = (v_1, v_2, \ldots, v_n)$ in n dimensions.

If v is composed of real components, its magnitude is defined as

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}\ .$$

As an example, consider the vector $\vec{a}$ represented by the line segment which goes from the point (1, 2) to the point (4, 6). Can you calculate the coordinates and the magnitude of this vector?

To find the coordinates, translate the line segment one unit left and two units down. The line segment begins at the origin and ends at $(4 - 1, 6 - 2) = (3, 4)$.

Therefore, $\vec{a} = (3, 4)$. The magnitude of $\vec{a}$ is $|\vec{a}| = \sqrt{3^2 + 4^2} = 5$.

The vector of sum is easy to express in terms of these coordinates. If $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$, their sum is simply $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2)$.

It is also easy to see that $\vec{b} - \vec{a} = (b_1 - a_1, b_2 - a_2)$ and $k\vec{a} = (ka_1, ka_2)$ for any scalar k.

Another way to denote vectors is in terms of the standard unit vectors (basis) denoted $\vec{i}$ and $\vec{j}$. A unit vector is a vector whose magnitude is one. The vector $\vec{i}$ is the unit vector in the direction of the positive x-axis. In coordinates, we can write $\vec{i} = (1, 0)$. Similarly, the vector $\vec{j}$ is the unit vector in the direction of the positive y-axis: $\vec{j} = (0, 1)$. We can write any two-dimensional vector in terms of these unit vectors as $\vec{a} = (a_1, a_2) = a_1\vec{i} + a_2\vec{j}$.

§ 16 VECTORS IN THREE-DIMENSIONAL SPACE

In three-dimensional space, there is a standard Cartesian coordinate system (x, y, z). Starting with a point which we call the origin, construct three mutually perpendicular axes, which we call the x-axis, the y-axis, and the z-axis.

With these axes any point p in space can be assigned three coordinates $\vec{p} = (p_1, p_2, p_3)$.

We often write $\vec{a} \in R^3$ to denote that it can be described by three real coordinates. Sums, differences and scalar multiples of three-dimensional vectors are all performed on each component.

If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3),$$

$$\vec{b} - \vec{a} = (b_1 - a_1, b_2 - a_2, b_3 - a_3),$$

$$k\vec{a} = (ka_1, ka_2, ka_3).$$

Just as in two dimensions, we can also denote three-dimensional vectors is in terms of the standard unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$. These vectors are the unit vectors in the positive *x*, *y*, and *z* direction, respectively.

In terms of coordinates, we can write them as $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$. We can express any three-dimensional vector as a sum of scalar multiples of these unit vectors in the form

$$\vec{a} = (a_1, a_2, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}.$$

Example.

Find the magnitude of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ by the coordinates of points *A*, *B* and *C* for the indicated vectors.

$A(-2, 3, -4)$, $B(3, -1, 2)$, $C(4, 2, 4)$, $\vec{a} = \overrightarrow{AC} + \overrightarrow{CB}$, $\vec{b} = \overrightarrow{AB}$, $\vec{c} = \overrightarrow{CB}$.

TASK TO REVIEW

Find the length of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ by the coordinates of points *A*, *B* and *C* for the indicated vectors.

1. $A(4, 6, 3)$, $B(-5, 2, 6)$, $C(4, -14, 0)$, $\vec{a} = \overrightarrow{CB} - \overrightarrow{AC}$, $\vec{b} = \overrightarrow{AB}$, $\vec{c} = \overrightarrow{CB}$.
2. $A(4, 3, -2)$, $B(-3, -1, 4)$, $C(2, 2, 1)$, $\vec{a} = \overrightarrow{AC} + \overrightarrow{CB}$, $\vec{b} = \overrightarrow{AB}$, $\vec{c} = \overrightarrow{AC}$.

3. $A(-2, -2, 4)$, $B(1, 3, -2)$, $C(1, 4, 2)$, $\vec{a} = \overrightarrow{AC} - \overrightarrow{BA}$, $\vec{b} = \overrightarrow{BC}$, $\vec{c} = \overrightarrow{BC}$.

4. $A(2, 0, 3)$, $B(3, 1, -4)$, $C(-1, 2, 2)$, $\vec{a} = \overrightarrow{BA} + \overrightarrow{AC}$, $\vec{b} = \overrightarrow{BA}$, $\vec{c} = \overrightarrow{BC}$.

5. $A(2, 4, 5)$, $B(1, -2, 3)$, $C(-1, -2, 4)$, $\vec{a} = \overrightarrow{AB} - \overrightarrow{AC}$, $\vec{b} = \overrightarrow{BC}$, $\vec{c} = \overrightarrow{AB}$.

6. $A(-1, -2, 4)$, $B(-1, 3, 5)$, $C(1, 4, 2)$, $\vec{a} = \overrightarrow{BA} + \overrightarrow{AC}$, $\vec{b} = \overrightarrow{AB}$, $\vec{c} = \overrightarrow{AC}$.

7. $A(1, 3, 2)$, $B(2, 4, 1)$, $C(1, 3, 2)$, $\vec{a} = \overrightarrow{AB} + \overrightarrow{CB}$, $\vec{b} = \overrightarrow{AC}$, $\vec{c} = \overrightarrow{AB}$.

8. $A(4, 0, -2)$, $B(3, -1, 4)$, $C(2, 2, 1)$, $\vec{a} = \overrightarrow{AC} + \overrightarrow{CB}$, $\vec{b} = \overrightarrow{AB}$, $\vec{c} = \overrightarrow{AC}$.

9. $A(-2, -2, 4)$, $B(10, 3, -2)$, $C(1, 0, 2)$, $\vec{a} = \overrightarrow{AC} - \overrightarrow{BA}$, $\vec{b} = \overrightarrow{BC}$, $\vec{c} = \overrightarrow{BC}$.

10. $A(2, 0, 3)$, $B(3, 1, -4)$, $C(-1, 8, 2)$, $\vec{a} = \overrightarrow{BA} + \overrightarrow{AC}$, $\vec{b} = \overrightarrow{BA}$, $\vec{c} = \overrightarrow{BC}$.

§ 17 DOT PRODUCT

The most important product of two vectors is their “dot product”, or “scalar product”. This product results in a scalar.

Vector dot product is equal to the sum of the products of the corresponding coordinates:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n.$$

We have another definition for the dot product:

$$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos\varphi,$$

where φ – angle between vectors.

Example.

Calculate the dot product of $\vec{a} = (1, 2, 3)$ and $\vec{b} = (4, -5, 6)$. Do the vectors form an acute angle, right angle, or obtuse angle?

Example.

Calculate the dot product of $\vec{a} = (-4, -9)$ and $\vec{b} = (-1, 2)$. Do the vectors form an acute angle, right angle, or obtuse angle?

Example.

If $\vec{a}$ = (6, −1, 3), for what value of c is the vector $\vec{b}$ = (4, c, −2) perpendicular to vector $\vec{a}$?

Properties of dot product:

1. $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$ (commutative property).
2. $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$ (distributive property).
3. $c(\vec{u}\cdot\vec{v})=(c\vec{u})\cdot\vec{v}=\vec{u}\cdot(c\vec{v})$ (associative property).
4. $\vec{v}\cdot\vec{v}=|\vec{v}|^2$ (property of magnitude).

Example.

Find the measure of the angle between each pair of vectors.

a) $\vec{i}+\vec{j}+\vec{k}$ and $2\vec{i}-\vec{j}-3\vec{k}$.

b) (2, 5, 6) and (−2, −4, 4).

Orthogonality criterion.

The nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal vector if and only if $\vec{u}\cdot\vec{v}=0$.

Example.

For which value of x is $\vec{p}=(2,8,-1)$ orthogonal to $\vec{q}=(-x,-1,2)$?

The vector projection of $\vec{v}$ onto $\vec{u}$ is the vector labeled $proj_{\vec{u}}\vec{v}$. If φ represents the angle between $\vec{u}$ and $\vec{v}$, then we have:

$$|\,proj_{\vec{u}}\vec{v}\,|=\frac{\vec{v}\cdot\vec{v}}{|\vec{u}|}.$$

Example.

Find the projection of vector $\vec{v}$ onto vector $\vec{u}$.

a) $\vec{v}=(3,5,1)$ and $\vec{u}=(-1,4,3)$.

b) $\vec{v}=3\vec{i}-2\vec{j}$ and $\vec{u}=\vec{i}+6\vec{j}$.

When a constant force is applied to an object so the object moves in a straight line from point P to point Q, the work W done by the force $\vec{F}$, acting at an angle φ from the line of motion, is given $W=\vec{F}\cdot\overrightarrow{PQ}$.

Example.

Suppose a child is pulling a wagon with a force having a magnitude of 8 on the handle at an angle of 60 degree. If the child pulls the wagon 50, find the work done by the force.

Example.

A conveyor belt generates a force $\vec{F} = 5\vec{i} - 3\vec{j} + \vec{k}$, that moves a suitcase from point (1, 1, 1) to point (9, 4, 7) along a straight line. Find the work done by the conveyor belt.

Example.

Find the scalar product of vectors $\vec{a}$ and $\vec{b}$ by the coordinates of points A, B and C for the indicated vectors; the projection of the vector $\vec{c}$ onto the vector $\vec{d}$.

1. $A(4, 6, 3)$, $B(-5, 2, 6)$, $C(4, -4, -3)$,
$\vec{a} = \overrightarrow{CB} - \overrightarrow{AC}$, $\vec{b} = \overrightarrow{AB}$, $\vec{c} = \overrightarrow{CB}$, $\vec{d} = \overrightarrow{AC}$.

2. $A(4, 3, -2)$, $B(-3, -1, 4)$, $C(2, 2, 1)$,
$\vec{a} = \overrightarrow{AC} + \overrightarrow{CB}$, $\vec{b} = \overrightarrow{AB}$, $\vec{c} = \overrightarrow{AC}$, $\vec{d} = \overrightarrow{CB}$.

3. $A(-2, -2, 4)$, $B(1, 3, -2)$, $C(1, 4, 2)$,
$\vec{a} = \overrightarrow{AC} - \overrightarrow{BA}$, $\vec{b} = \overrightarrow{BC}$, $\vec{c} = \overrightarrow{BC}$, $\vec{d} = \overrightarrow{AC}$.

4. $A(2, 4, 3)$, $B(3, 1, -4)$, $C(-1, 2, 2)$,
$\vec{a} = \overrightarrow{BA} + \overrightarrow{AC}$, $\vec{b} = \overrightarrow{BA}$, $\vec{c} = \vec{b}$, $\vec{d} = \overrightarrow{AC}$.

5. $A(2, 4, 5)$, $B(1, -2, 3)$, $C(-1, -2, 4)$,
$\vec{a} = \overrightarrow{AB} - \overrightarrow{AC}$, $\vec{b} = \overrightarrow{BC}$, $\vec{c} = \vec{b}$, $\vec{d} = \overrightarrow{AB}$.

6. $A(-1, -2, 4)$, $B(-1, 3, 5)$, $C(1, 4, 2)$,
$\vec{a} = \overrightarrow{AC} - \overrightarrow{BC}$, $\vec{c} = \vec{b} = \overrightarrow{AB}$, $\vec{d} = \overrightarrow{AC}$.

7. $A(-2, -2, 4)$, $B(0, 3, 2)$, $C(1, 4, 2)$,
$\vec{a} = \overrightarrow{AC} - \overrightarrow{BA}$, $\vec{b} = \overrightarrow{BC}$, $\vec{c} = \overrightarrow{BC}$, $\vec{d} = \overrightarrow{AC}$.

8. $A(2, 4, 3)$, $B(3, 1, 4)$, $C(-1, 2, 2)$,

$\vec{a} = \overrightarrow{BA} + \overrightarrow{AC}$, $\vec{b} = \overrightarrow{BA}$, $\vec{c} = \vec{b}$, $\vec{d} = \overrightarrow{AC}$.

9. $A(2, 9, 5)$, $B(1, 2, 3)$, $C(-1, -2, 4)$,

$\vec{a} = \overrightarrow{AB} - \overrightarrow{AC}$, $\vec{b} = \overrightarrow{BC}$, $\vec{c} = \vec{b}$, $\vec{d} = \overrightarrow{AB}$.

10. $A(1, -2, 4)$, $B(1, 3, 5)$, $C(1, 4, 2)$,

$\vec{a} = \overrightarrow{AC} - \overrightarrow{BC}$, $\vec{c} = \vec{b} = \overrightarrow{AB}$, $\vec{d} = \overrightarrow{AC}$.

§ 18 CROSS PRODUCT

There are two ways to take the product of a pair of vectors. One of these methods of multiplication is the cross product.

The cross product is defined only for three-dimensional vectors. If $\vec{a}$ and $\vec{b}$ are two three-dimensional vectors, then their cross product written as $\vec{a} \times \vec{b}$ (“a cross b”) is another three-dimensional vector.

We define this cross product vector $\vec{a} \times \vec{b}$ by the following three requirements:

1. $\vec{a} \times \vec{b}$ is a vector that is perpendicular to both $\vec{a}$ and $\vec{b}$.

2. The magnitude of the vector $\vec{a} \times \vec{b}$ written as $|\vec{a} \times \vec{b}|$ is the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin \theta,$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

3. The direction of $\vec{a} \times \vec{b}$ is determined by the right-hand rule. This means that if we curl the fingers of the right hand from $\vec{a}$ to $\vec{b}$, then the thumb points in the direction of $\vec{a} \times \vec{b}$.

If $\vec{a}$ and $\vec{b}$ are parallel or if either vector is the zero vector, then the cross product is the zero vector.

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$.

Then, the cross product

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.$$

Example.

Find a cross product $\vec{p}\times\vec{q}$ and $\vec{q}\times\vec{p}$ of $\vec{p}$ = (−1, 2, 5) and $\vec{q}$ = (4, 0, −3).

As we have seen, the dot product is often called the scalar product because it results in a scalar. The cross product results in a vector, so it is sometimes called the vector product.

Cross product is anticommutativity:

$$\vec{p}\times\vec{q}\neq\vec{q}\times\vec{p}\ .$$

Let’s explore some properties of the cross product.

1. Anticommutative property:

$$\vec{u}\times\vec{v}=-\vec{v}\times\vec{u}\ .$$

2. Distributive property:

$$\vec{u}\times(\vec{v}+\vec{w})=\vec{u}\times\vec{v}+\vec{u}\times\vec{w}\,.$$

3. Multiplication by a constant:

$$c(\vec{u}\times\vec{v})=(c\vec{u})\times\vec{v}=\vec{u}\times(c\vec{v})\ .$$

4. Cross product of the zero vector:

$$\vec{u}\times\vec{0}=\vec{0}\times\vec{u}=\vec{0}\,.$$

5. Cross product of a vector with itself:

$$\vec{u}\times\vec{u}=\vec{0}\,.$$

To use the cross product for calculating areas, we state and prove the following theorem.

Area of a parallelogram. If we locate vectors $\vec{p}$ and $\vec{q}$ such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by $|\vec{p}\times\vec{q}|$.

Example.

Let $P = (1, 0, 0)$, $Q = (0, 1, 0)$, and $R = (0, 0, 1)$ be the vertices of a triangle. Find its area.

Example.

Find the area of the parallelogram $PQRS$ with vertices $P(1, 1, 0)$, $Q(7, 1, 0)$, $R(9, 4, 2)$, and $S(3, 4, 2)$.

Torque T measures the tendency of a force to produce rotation about an axis of rotation. Let $\vec{r}$ be a vector with an initial point located on the axis of rotation and with a terminal point located at the point where the force is applied, and let vector $\vec{F}$ represent the force. Then torque is equal to the cross product of $\vec{r}$ and $\vec{F}$:

$$\vec{T}=\vec{r}\times\vec{F}.$$

Example.

Vectors are given. It is necessary:

a) find the magnitude of $\vec{a}$;

b) the vector product of vectors $\vec{a}$ and $\vec{b}$;

c) check whether two vectors and are collinear or orthogonal $\vec{a}$ and $\vec{c}$.

1. $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$, $\vec{b} = \vec{j} + 4\vec{k}$, $\vec{c} = 5\vec{i} + 2\vec{j} - 3\vec{k}$.
2. $\vec{a} = 3\vec{i} + 4\vec{j} + \vec{k}$, $\vec{b} = \vec{i} - 2\vec{j} + 7\vec{k}$, $\vec{c} = 3\vec{i} - 6\vec{j} + 21\vec{k}$.
3. $\vec{a} = 2\vec{i} - 4\vec{j} - 2\vec{k}$, $\vec{b} = 7\vec{i} + 3\vec{j}$, $\vec{c} = 3\vec{i} + 5\vec{j} - 7\vec{k}$.
4. $\vec{a} = -7\vec{i} + 2\vec{k}$, $\vec{b} = 2\vec{i} - 6\vec{j} + 4\vec{k}$, $\vec{c} = -3\vec{j} + 2\vec{k}$.
5. $\vec{a} = 5\vec{i} - 3\vec{j} + \vec{k}$, $\vec{b} = 5\vec{j} + 4\vec{k}$, $\vec{c} = 5\vec{i} + 2\vec{j} - 3\vec{k}$.
6. $\vec{a} = \vec{i} + 4\vec{j} + \vec{k}$, $\vec{b} = \vec{i} - 8\vec{j} + 7\vec{k}$, $\vec{c} = 3\vec{i} - 6\vec{j} + 21\vec{k}$.

7. $\vec{a} = 2\vec{i} - 4\vec{j} - 2\vec{k}$, $\vec{b} = 7\vec{i} + 9\vec{j}$, $\vec{c} = 3\vec{i} + 5\vec{j} - 7\vec{k}$.

8. $\vec{a} = 7\vec{i} + 2\vec{k}$, $\vec{b} = 2\vec{i} - 6\vec{j} + 4\vec{k}$, $\vec{c} = -3\vec{j} + 2\vec{k}$.

9. $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$, $\vec{b} = \vec{j} + 4\vec{k}$, $\vec{c} = 2\vec{j} - 3\vec{k}$.

10. $\vec{a} = -4\vec{j} - 2\vec{k}$, $\vec{b} = 7\vec{i} + 3\vec{j}$, $\vec{c} = 3\vec{i} + 5\vec{j} - 7\vec{k}$.

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