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MEASUREMENT ERRORS.
ANALYSIS AND EVALUATION

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The manual for the course of physics is intended to familiarize 1st year students with the basics of the theory of errors, as well as methods for calculating errors that are necessary when performing laboratory works. The manual presents theoretical material on the analysis and methods for calculating the errors of direct and indirect measurements. In addition, methodological materials are given for performing laboratory work, the purpose of which is to consolidate the skill of conducting precision measurements and processing their results.

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1. MEASUREMENTS

One of the main tasks of any physical experiment is the measurement of different physical quantities.

Physical quantity is a property of a physical object or phenomenon that can be quantified by measurement.

Measurement consists in comparing the required physical quantity with another quantity taken as a standard or *unit*.

All the measurements could be considered as either direct or indirect. *Direct measurement* occurs when you take a measurement explicitly of the object's characteristic you want to measure. For instance, you can directly measure an object length with a measuring tape, room temperature with a thermometer, electrical current with an amperemeter, or an object weight with a chemical balance. In contrast, *indirect measurement* occurs when you measure something else and then convert the obtained quantities according to some mathematical expressions into a measurement of the characteristic in question. As an example, measurement of an object density is indirect: it consists in a preliminary measurement of a mass and volume of a physical object and subsequent calculation of the object density as a ratio of the measured mass and volume.

Any measurement gives a result that is somewhat different from the true value of the measured quantity. Often, the very concept of the “true value” of a physical quantity does not exist in nature. For example, if we measure the diameter of a wire, then we will encounter the fact that the cross section of the wire is not perfectly round, but is an ellipse, the parameters of which, moreover, change slightly from one section of the wire to another. Therefore, in practice, the actually solved problem is determining the *mean value* of the measured quantity which would most accurately characterize the object under study.

The result of the observation of a physical quantity is a *reading*, giving one value of the measured quantity. The measurement result, as a set of actions, is obtained after mathematical processing of all the readings.

A *measurement with single observations* is a measurement in which each reading is taken at different values of other physical quantities associated with the measurand.

A *measurement with multiple observations* is a measurement in which all readings are obtained at fixed values of other physical quantities associated with the measurand.

When directly measuring a physical quantity, three consecutive operations have to be performed:

1. Choice of the device, its verification and adjustment.
2. Observation of instrument readings (for multiple observations, the operation will be repeated several times).
3. Calculation of the mean value of a physical quantity from the measurement results and error estimation.

2. ERRORS OF MEASUREMENTS

Any measurement is accompanied by errors. Each measurement gives the value of the determined quantity x with some absolute error Δx . This means that the true value lies in the interval

$$\langle x \rangle - \Delta x \leq x \leq \langle x \rangle + \Delta x, \quad (2.1)$$

where $\langle x \rangle$ is the mean value of the quantity obtained as the closest to the true when measured.

The interval of values $[\langle x \rangle - \Delta x; \langle x \rangle + \Delta x]$ which contains the true value of the measured quantity with a certain degree of reliability (confidence probability p) is called the *confidence interval*.

The *confidence probability* p is the probability with which the confidence interval includes the true value of the measured value.

Thus, the error is characterized by a confidence interval (Fig. 2.1).

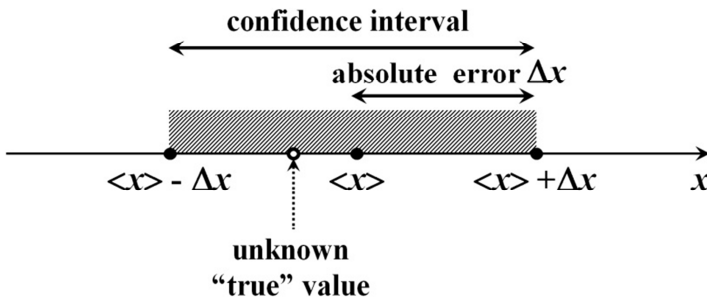


Fig. 2.1. Measurement result

The smaller is the confidence interval, the higher is the accuracy of the measurement.

The measurement error can be represented either as an absolute error or as a relative error.

The *absolute error* Δx is equal to the modulus of difference between the boundary of the confidence interval and the mean value, i. e. is the half-width of the confidence interval. The absolute error is measured in the same units as the quantity itself.

The *relative error* shows the proportion taken by the error from the mean value of the quantity. It is equal to the ratio of the absolute error to the mean value of a given physical quantity and is usually expressed as a percentage:

$$\varepsilon_x = \frac{\Delta x}{\langle x \rangle} 100 \%. \quad (2.2)$$

The most complete representation of the measurement result is as follows:

$$x = \langle x \rangle \pm \Delta x, \text{ units}; \quad \varepsilon_x = \frac{\Delta x}{\langle x \rangle} 100 \%. \quad (2.3)$$

The reciprocal of the relative error, $\frac{\langle x \rangle}{\Delta x}$, is called *accuracy*. The smaller is the relative error, the higher is the measurement accuracy.

To compare the results of measurements of the same physical quantity, carried out using different instruments or different methods, one should check the overlap of the corresponding confidence intervals. If the confidence intervals do overlap, then the measurement results are in agreement within the error. However, if the measurements do not have common values, then the results are inconsistent (Fig. 2.2).

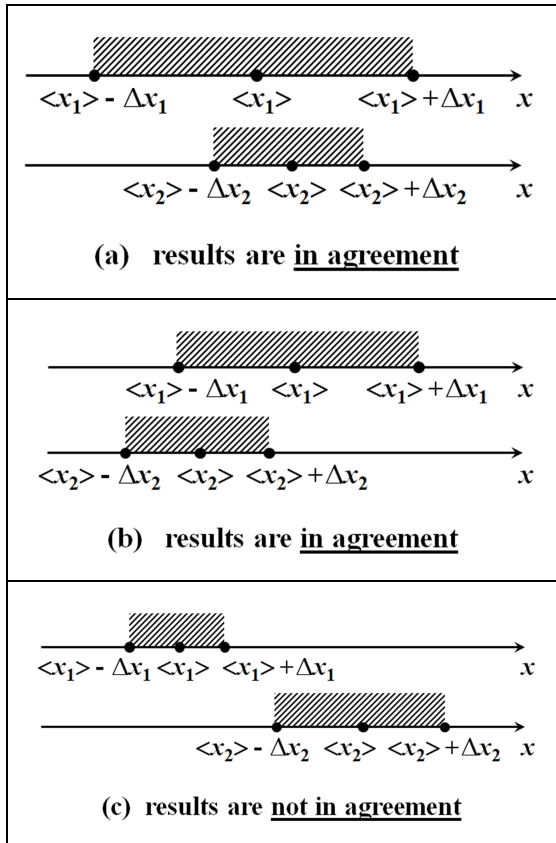


Fig. 2.2. Comparing the results of two measurements of the same value

3. ANALYSIS OF ERRORS

Summarizing the above, it is impossible to absolutely accurately determine the true value of a physical quantity. The reasons for the occurrence of the measurement errors are different and may be due to the characteristics of the object under study, measuring instruments and other reasons.

According to the impact on the measurement result, the following types of errors can be distinguished (Fig. 3.1).

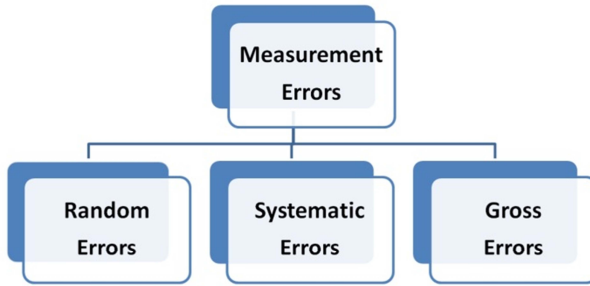


Fig. 3.1. Types of errors classified by impact on results

Random error is an error that changes randomly when measurements are repeated.

Systematic error is an error that remains constant when measurements are repeated.

Gross error is an error that significantly exceeds what is expected under given conditions.

According to the sources of errors, the following types of errors are distinguished (Fig. 3.2).

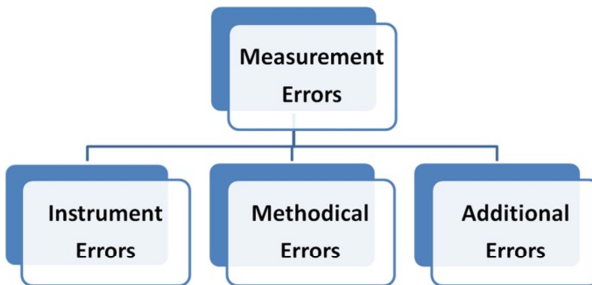


Fig. 3.2. Types of errors classified by their origin

Instrumental error is an error of measuring instruments.

Methodological error is an error due to the imperfection of the measurement method. Methodical errors include errors that are due to the accepted simplifications and assumptions in the measurement of the model, the inaccuracy of the empirical formulas used in the calculations, the limited accuracy of the constants used in the equations, and the like.

Additional error is an error due to the influence of factors that are not taken into account in the model of the measurement object. Such factors can include both environmental factors (temperature, moisture, dirt, vibration, electrostatic or magnetic fields, etc.), and the human factor (imperfection of the operator's senses or his inattention).

In general, all three of these sources of errors can give both systematic and random contributions into the total measurement error. The specific contributions of these components depend on the organization of the experiment.

When carrying out *measurements in the educational laboratory*, the experiment is organized in such a way that:

1. The instrument error has only a systematic component.
2. The additional error has only a random component.
3. Methodical error can be neglected.
4. The accuracy of instrument readings is guaranteed.

3.1. Random Errors

Random errors are the measurement errors which are randomly changing from count to count.

Random errors may change the result both up and down with the same probability, and the larger is the error, the less often it appears in the experiment.

Random errors are due to irregularly changing experimental conditions (temperature changes, ground vibrations, disturbance of the air, and the like), poor repeatability of the instrument readings, imperfection of the experimenter's senses, etc.

It is impossible to eliminate a random measurement error, but it can be mitigated by increasing the number of readings (experiment repetitions).

3.2. Systematic Errors

Systematic errors are the measurement errors whose value does not change from experiment to experiment.

A systematic error changes the result of the experiment always in the same direction and by the same amount, i. e. it is constant in sign and magnitude.

Systematic errors may be due to a malfunction or incorrect adjustment of the measuring device (inaccurate breakdown of the scale of the ruler, inaccurate weight of calibration weights, slow stopwatch, etc.), as well as features of the object under study and do not depend on the number of measurements.

Taking into account and excluding (or reducing) the systematic error is one of the most difficult problems in the theory of measurements. Methods for solving this problem depend on specific types of measurements, and there is no some general methodology for solving it.

A systematic error due to inaccuracy or malfunction of the measuring instrument can be established by checking the instrument. If a systematic error caused by the device is detected, you can:

- 1) replace the device with a serviceable one;
- 2) adjust or repair the device;
- 3) determine the sign and magnitude of the error and introduce a correction;
- 4) in some cases, it is possible to change the measurement procedure in order to eliminate the systematic error.

For accurate measurements, the systematic error is estimated based on the results of measuring the desired value by various fundamentally independent methods using different equipment.

3.3. Instrument Errors

Instrument error is an error due to the limited accuracy of the used instrument.

For example, using an ordinary millimeter ruler, one cannot achieve micron accuracy, as well as using ordinary wristwatches, it is impossible to measure time with an accuracy of up to a millisecond. That is, the use of a particular instrument determines the maximum accuracy that can be achieved with its help.

The instrument error can be reduced by using a device of a higher accuracy class (for example, instead of a ruler, take a caliper; if this is not enough, take a micrometer, etc.).

3.4. Gross Errors

Gross Errors are abnormally large errors that completely distort the measurement result.

Gross errors are caused by abrupt changes in the experimental conditions (for example, power surges in the electrical network) or inattentive actions of the experimenter.

Results containing gross errors must be identified and excluded from consideration. In practice, for a more reliable elimination of gross errors, several additional measurements are often specially made, and the largest and smallest values are discarded before starting the processing of the results.

4. EVALUATION OF ERRORS FOR DIRECT MEASUREMENTS

4.1. Evaluation of Random Errors

Let us consider a direct measurement of the value x , in which n readings were obtained: $x_1, x_2, x_3, \dots, x_n$.

In the presence of random errors, the observed values of the measured quantity are randomly scattered relative to its true value. In this case, the best value is found as the most probable of a series of readings, and the error is characterized by the width of the confidence interval, which includes the true value with a given confidence probability.

The best estimate of the true value of x is the sample *mean value* $\langle x \rangle$, which can be calculated as the arithmetic mean:

$$\langle x \rangle = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{1}{n} \sum_{i=1}^n x_i, \quad (4.1)$$

where x_i is the i -th reading of value x .

To estimate the scatter of readings relative to the best value, the *standard deviation (standard error)* from the mean value is used:

$$\sigma = \sqrt{\frac{1}{n(n-1)} \sum_{i=1}^n (\langle x \rangle - x_i)^2}. \quad (4.2)$$

Then the *random error* of multiple measurements Δx_r will be calculated by the formula:

$$\Delta x_r = t_{n,p} \sigma. \quad (4.3)$$

Here $t_{n,p}$ is a *Student's coefficient* (dimensionless confidence coefficient), which shows how many times it is necessary to increase the standard deviation from the mean value in order to obtain a given reliability of the result with a given number of readings. The Student's coefficient depends in a complex way on the reliability (confidence probability) p and the number of measurements n . Its value can be determined from statistical tables (see Appendix C).

Note that when performing laboratory work in a physical workshop, the standard confidence probability is $p = 0.95$.

The greater the confidence probability, the more reliable the estimate of the confidence interval and, at the same time, the wider its boundaries.

In the simplest case, one can calculate a random error as modulo mean deviation of the readings from the mean value $\langle x \rangle$:

$$\Delta x_r = \frac{1}{n} \sum_{i=1}^n |\langle x \rangle - x_i|. \quad (4.4)$$

This estimate of the random error is not strict, it gives a slightly overestimated error value and does not allow one to accurately estimate the probability with which the sought value should fall within the confidence interval, but due to its simplicity it is used quite often.

4.2. Evaluation of Instrument Errors

To characterize measuring instruments, the concept of accuracy class is often used.

The accuracy class is the ratio of the absolute instrument error Δx_a to the limit value x_{\max} of the measured quantity (its highest value that can be measured on the scale of the instrument):

$$k = \frac{\Delta x_a}{x_{\max}} 100. \quad (4.5)$$

There are 7 classes of instrument accuracy: 0.1; 0.2; 0.5; 1.0; 1.5; 2.5; 4.0. Instruments of accuracy class 0.1; 0.2; 0.5 is used for accurate laboratory measurements. They are called *precision instruments*. For technical measurements, one can use instruments of classes 1.0; 1.5; 2.5; 4.0.

The accuracy class of the device is indicated on the scale of the device. If there is no such designation on the scale, then this device is out of class.

Having determined the accuracy class k and the limit value on the scale of the instrument, it is easy to calculate its absolute instrument error:

$$\Delta x_a = k \frac{x_{\max}}{100}. \quad (4.6)$$

For optimal use of the device, one should select the device so that the measured value falls at the end of the scale of the device. This provides the smallest relative error.

If the accuracy class of the device is unknown, then the instrument error is taken equal to half of the scale division value:

$$\Delta x_a = \frac{c}{2}. \quad (4.7)$$

Scale division value c is a distance between adjacent scale marks.

If the device has an additional vernier scale that increases the measurement accuracy, then the instrument error is estimated as half scale division value of the vernier scale.

4.3. Evaluation of Total Error

The *total absolute error* of direct measurements is equal to the quadratic sum of all its components

$$\Delta x = \sqrt{\sum_m (\Delta x_m)^2}. \quad (4.9)$$

Here m is the number of different contributions to the total error.

As it was discussed above, when performing laboratory work of a physical practicum, all contributions to the error can be neglected, except for random error Δx_r and instrument error Δx_a . So the total absolute error of direct measurement will be calculated by the formula:

$$\Delta x = \sqrt{(\Delta x_r)^2 + (\Delta x_a)^2}. \quad (4.10)$$

4.4. Rounding of Errors

Significant digits (figures) are the digits of a number that are meaningful in terms of accuracy.

Significant digits are counted starting the first non-zero digit in a number, but they do include all the subsequent zero (and obviously non-zero) digits. Importantly, all zeros at the beginning of a number (does not matter before or after the decimal point) are not significant. All zeros to the power of ten in the standard number representation are not significant (Fig. D.1 in Appendix D).

For example, in number 0.0540 there are 3 significant digits: 5, 4 and last 0. In number $1.037 \cdot 10^5$ there are 4 significant digits: 1, 0, 3 and 7.

All the significant digits can be divided into three groups: correct, doubtful and incorrect digits. A *doubtful digit* of a number is a digit in the position that corresponds to the position of the first significant digit in the absolute error value. For example, $(2.43 \pm 0.07) \cdot 10^3$, here 7 is the first significant digit in the absolute error and 3 is the doubtful digit in the mean value.

All the digits placed on the left from the doubtful digit are called *correct digits*.

All the digits placed on the right from the doubtful digit are called *incorrect digits*. It makes no sense to provide such incorrect digits and it is recommended to exclude them by rounding the mean value to a doubtful digit according to the rounding rules.

The absolute errors have no correct digits, the first significant digit of an absolute error is already doubtful. All other significant figures in the absolute error are incorrect and in the educational laboratory they must be excluded from the reported result by *rounding up* the first significant figure.

However, to avoid unnecessary coarsening of the final result, if the first significant digit in the absolute error is "1" or "2", there remains one more significant digit.

After correctly rounding first the error and then the mean value, *the number of decimal places in the absolute error and in the mean value should be the same.*

Here are some examples of the correct recording of the final result: 9.5 ± 0.8 ; 2.350 ± 0.013 ; $(5.6 \pm 0.4) \cdot 10^{-2}$; $(1.03 \pm 0.25) \cdot 10^5$.

4.5. Algorithms for Processing Direct Measurements

Basic algorithm:

1. Determine the instrument error Δx_a (Eq. 4.7 or 4.6).
2. Calculate the mean value of the measurement $\langle x \rangle$ (Eq. 4.1).
3. Calculate the standard deviation of the mean value σ (Eq. 4.2).
4. Determine the Student's coefficient $t_{n,p}$ for the given reliability $p = 0.95$ and the obtained number of readings n (Appendix C).
5. Calculate the random error Δx_r (Eq. 4.3).
6. Calculate the total error Δx (Eq. 4.10).
7. After rounding, write the result of processing measurements in the form:

$$x = \langle x \rangle \pm \Delta x \text{ units}; \quad \varepsilon_x = \frac{\Delta x}{\langle x \rangle} 100 \%$$

Where "units" corresponds to the required units, for example, "s" for seconds.

Simplified algorithm:

1. Determine the instrument error Δx_a (Eq. 4.7 or 4.6).
2. Calculate the mean value of the measurement $\langle x \rangle$ (Eq. 4.1).
3. Calculate the random error Δx_r as the modulo average deviation of the measurement result from the mean value $\langle x \rangle$ (Eq. 4.4).
4. Calculate the total error Δx (Eq. 4.10).

5. After rounding, write the result of processing measurements in the form:

$$x = \langle x \rangle \pm \Delta x \text{ units}; \quad \varepsilon_x = \frac{\Delta x}{\langle x \rangle} 100 \%$$

Examples of recording the final result:

$$t = 5.08 \pm 0.14 \text{ s}; \quad \varepsilon_t = 2.8 \%$$

$$L = (6.4 \pm 0.7) \cdot 10^{-3} \text{ m}; \quad \varepsilon_L = 11 \%$$

5. EVALUATION OF ERRORS FOR INDIRECT MEASUREMENTS

5.1. Evaluation of Errors for Functions of Several Variables

Let be $F = F(x, y, \dots)$ a functional relationship between the indirectly measured value F and the variables x and y , the values of which are found by direct measurements ($x = \langle x \rangle \pm \Delta x$, $y = \langle y \rangle \pm \Delta y, \dots$).

The mean value of the function $\langle F \rangle$ is determined by the mean values of the variables x and y :

$$\langle F \rangle = F(\langle x \rangle, \langle y \rangle, \dots). \quad (5.1)$$

Now we can obtain an expression for the absolute error ΔF . If we fix the values of all arguments except one, for example x , then the increment of the function when its argument changes has the form:

$$\Delta F_x = F(\langle x \rangle + \Delta x, \langle y \rangle, \dots) - F(\langle x \rangle, \langle y \rangle, \dots). \quad (5.2)$$

If the value Δx is small, then the function $F = F(x)$ can be considered as linear in the interval $[\langle x \rangle - \Delta x; \langle x \rangle + \Delta x]$ and the error ΔF_x can be expressed in terms of the partial derivative of the function F with respect to x :

$$\Delta F_x = \frac{\partial F}{\partial x} \Delta x. \quad (5.3)$$

The value ΔF_x describes the error ΔF due to the error Δx . Similarly, the contribution ΔF_y due to the error Δy and the contributions of the remaining arguments in ΔF are determined.

The *total absolute error* ΔF of indirect measurement F is calculated either using quadratic summation or modulo summation of its components introduced by each argument:

$$\Delta F = \sqrt{\left(\frac{\partial F}{\partial x} \Delta x\right)^2 + \left(\frac{\partial F}{\partial y} \Delta y\right)^2 + \dots}; \quad (5.4)$$

$$\Delta F = \left|\frac{\partial F}{\partial x}\right| \Delta x + \left|\frac{\partial F}{\partial y}\right| \Delta y + \dots \quad (5.5)$$

Relation (5.4) is applied when two conditions are met. First, the error of the arguments is due to the influence of many factors, among which there is no dominant factor. Secondly, the errors of the arguments are not statistically related.

In all other cases, one should use the relation (5.5). It often overestimates the errors of indirect measurements, however, when estimating errors, it is better to overestimate rather than underestimate.

The *relative error of indirect measurements* ε_F is calculated according to the definition:

$$\varepsilon_F = \frac{\Delta F}{\langle F \rangle}. \quad (5.6)$$

Appendix E contains expressions corresponding to the absolute and relative errors calculated using Eq. 5.5 for the most common functional dependencies in physical practice.

5.2. Algorithm for Processing Indirect Measurements

1. Using the known dependence of the indirectly measured quantity $F = F(x, y, \dots)$ on the arguments, determine the average value of the function $\langle F \rangle$ from the mean values of the arguments (Eq. 5.1).
2. Obtain an analytical expression for calculating the absolute error ΔF of the indirect measurement (Eq. 5.4 or 5.5, see also Appendix E).
3. Obtain an analytical expression for calculating the relative error ε_F of the indirect measurement (Eq. 5.6, see also Appendix E).
4. Estimate the resulting expressions for ΔF and ε_F by the complexity of calculations. Start calculations with the least time-consuming expression.
5. Using Eq. 5.6, calculate the missing error (ΔF or ε_F).
6. After rounding, write down the result of processing the indirect measurement in the form:

$$F = \langle F \rangle \pm \Delta F \text{ units}; \quad \varepsilon_F = \frac{\Delta F}{\langle F \rangle} 100 \%$$

6. LABORATORY WORK ON PROCESSING THE RESULTS OF MEASUREMENTS “MEASUREMENT OF BODY MASS BY INDIRECT METHOD”

Objectives:

1. To learn how to measure with a vernier caliper and micrometer.
2. To determine the mass of a wood cylinder by an indirect method, sequentially using three instruments: a ruler, a vernier caliper, a micrometer.
3. To study the method for estimating the errors of direct and indirect measurements.

Equipment:

1. Millimeter ruler.
2. Vernier caliper.
3. Micrometer.
4. Wood cylinder.

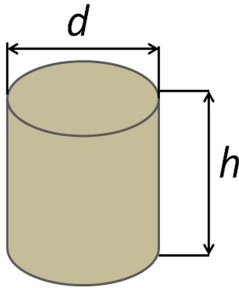
Procedure

Fig. 6.1. Cylinder parameters

In this work, we shall measure a mass of a given body by an indirect method. The studied body is a straight cylinder (Fig. 6.1) made from a solid piece of wood (specifically, dry beech). Therefore, its mass can be determined indirectly using the following calculation formula:

$$m = \rho V = \rho \pi \frac{d^2}{4} h. \quad (6.1)$$

Here, ρ is the density of the body material and $V = (\pi d^2 / 4)h$ is the body volume, calculated from the measured diameter d and height h of the cylinder.

Direct measurements are reduced to determining the linear dimensions of the cylinder (d and h). Then, using the tabular value of the density of dry beech $\rho = 650 \text{ kg/m}^3$, you need to calculate the mass of the cylinder. As a result of the experiment, it is necessary to find the mean value of the cylinder mass and evaluate the absolute and relative errors with which the cylinder mass was measured. The final result should be presented in the form:

$$m = \langle m \rangle \pm \Delta m. \quad (6.2)$$

6.1. Length Measurement Instruments

The most common instruments used to measure linear dimensions include a ruler, a vernier caliper and a micrometer. Each of them can be

used to measure the height and the diameter of our wood cylinder, but with a different accuracy.

Ruler

The simplest and the least accurate instrument is a ruler. It has a uniform scale along its straight measuring edge (Fig. 6.2). The distance between adjacent marks of the scale is called the scale division value c . Typically, the scale division value of the ruler $c = 1 \text{ mm}$, although the most accurate metal rulers often have a scale with $c = 0.5 \text{ mm}$.

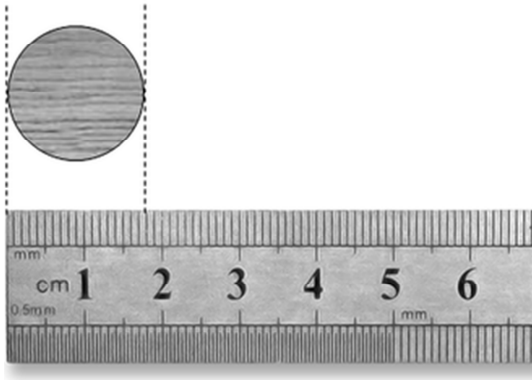


Fig. 6.2. Taking readings with a ruler

The length is measured with rulers in units of scale divisions covered, rounded up to the nearest integer number.

The instrument error is equal to half of the scale division value:

$$\Delta_a = \frac{c}{2} = \frac{1 \text{ mm}}{2} = 0.5 \text{ mm} \quad (\text{for a standard millimeter-scale ruler}).$$

Vernier Caliper

To measure linear dimensions with approximately 10 times higher precision, one should use a vernier caliper (Fig. 6.3). It was invented in 1631 by French scientist Pierre Vernier (1580–1637).

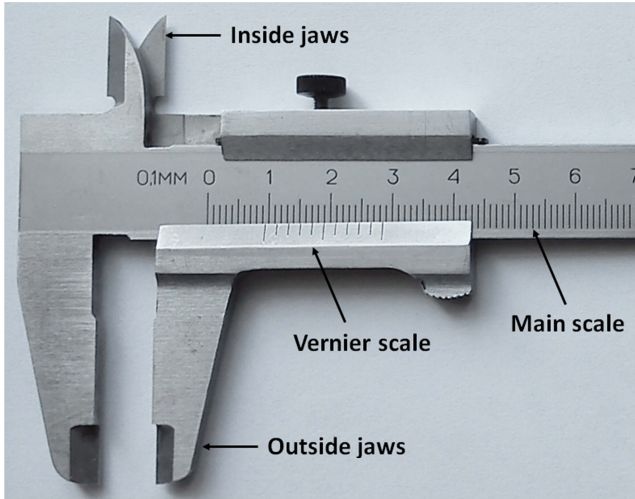


Fig. 6.3. Vernier caliper

The main idea of the vernier caliper is the usage of the second scale (vernier scale) that can move along the ruler with the main scale. The vernier scale is marked with divisions slightly smaller than the divisions of the main scale. For example, a vernier scale could have 11 markings for every 10 on the main scale. That is 10 divisions on the vernier scale correspond to 9 divisions on the main scale. This means that the vernier divisions are each 90 % of the main scale divisions. In general case, the value of a vernier scale division c_1 is equal to

$$c_1 = \frac{(N-1)}{N}c_0, \quad (6.3)$$

where c_0 is the value of main scale division, N is the number of divisions of the vernier scale. The number of divisions N determines the resulting accuracy of the vernier caliper.

The difference between the value of one main scale division and the value of one vernier scale division is known as the *least count* c of the vernier scale:

$$c = c_0 - c_1 = \frac{c_0}{N}. \quad (6.4)$$

For instance, for the caliper with $N = 10$ shown in Fig. 6.3, the least count c is equal to $1/10 = 0.1$ mm. For the most popular vernier calipers with $N = 20$, the least count c is equal to $1/20 = 0.05$ mm.

Connection between a main scale division and a vernier scale division is illustrated in Fig. 6.4.

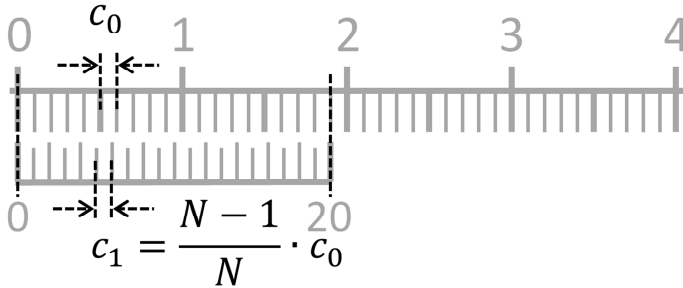


Fig. 6.4. Vernier caliper with $N = 20$: connection between a main scale division c_0 and a vernier scale division c_1

When the measured distance b is smaller than the value of main scale division c_0 (Fig. 6.5), it is evidently that

$$b = m(c_0 - c_1) = mc_0 / N = mc,$$

where m is the integer showing which of the marks on vernier scale is aligned with a mark on the main scale.

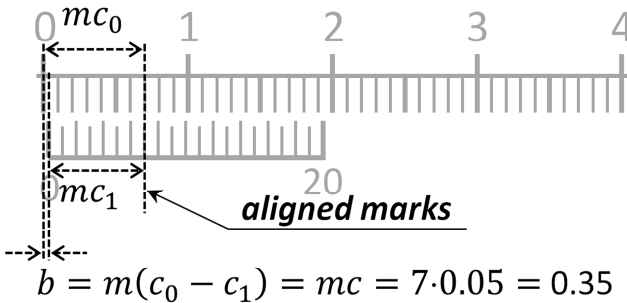


Fig. 6.5. Vernier caliper: about operating principle

When the measured distance x is larger than the value of main scale division c_0 (Fig. 6.6), then x is a sum of the readings of the main and vernier scales $x = a + b = a + mc$, where m is the same integer taken at the vernier scale, while a is the length taken in terms of the scale division values c_0 at the main scale.

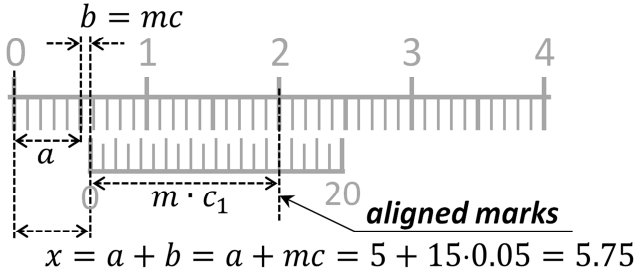


Fig. 6.6. Taking readings with a vernier caliper

The instrument error of a vernier caliper is taken equal to half of the least count value:

$$\Delta x_a = \frac{c}{2}.$$

For instance, for the calipers with $N = 10$,

$$\Delta_a = \frac{c}{2} = \frac{0.1 \text{ mm}}{2} = 0.05 \text{ mm}.$$

For the calipers with $N = 20$,

$$\Delta_a = \frac{c}{2} = \frac{0.05 \text{ mm}}{2} = 0.025 \text{ mm}.$$

It is important to note that the presence of a vernier scale introduces an additional source of errors, the so-called *zero error*. In particular, for a properly adjusted caliper, the first marks on both scales must be aligned with each other for zero-sized objects. Any discrepancy between them introduces an additional systematic error.

How to use the vernier caliper:

1. Check for the zero error: close the jaws and make sure that zero mark on the main scale and zero mark on the vernier scale are coincident (if not, then adjust the position of vernier scale).
2. Take the measured object and place it in between jaws:
 - use the outside jaws to measure external diameter or width of an object;
 - use the inside jaws to measure internal diameter of an object.
3. Take the reading: *Actual reading = Main scale + Vernier scale*:
 - *main scale reading*: zero mark on the vernier scale indicates the main scale reading (the nearest left integer);
 - *vernier scale reading*: a mark on the vernier scale, which strictly coincides with a mark on the main scale, gives a reading of the vernier scale.

Micrometer

To measure linear dimensions with even higher precision, you can use a micrometer caliper (Fig. 6.7). Although the first version of a micrometer (used with a telescope for precise astronomical observations) was invented around 1638 by an English astronomer William Gascoigne (1612–1644), its modern look and applications for measuring lengths of small objects originate from the middle of the XIX century only.

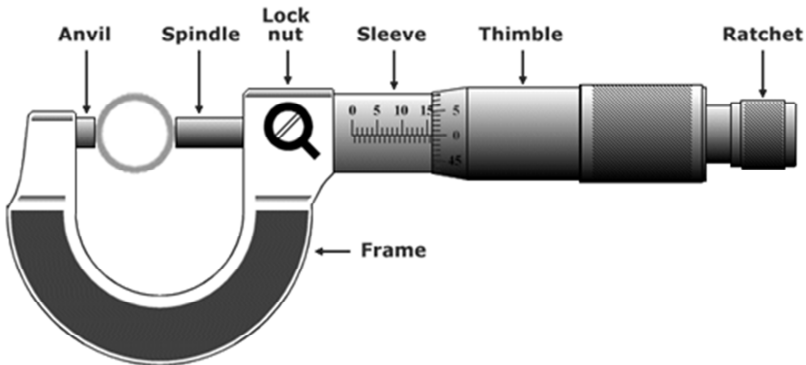


Fig. 6.7. Micrometer

A micrometer caliper (more often called simply a micrometer) consists of U-shaped frame fitted with an anvil at one side and a screwed spindle at the other side. The spindle passes through the frame and sleeve

with the main scale and is attached to a thimble with a ratchet and an additional rotating scale. In contrast to the vernier caliper, the micrometer works on the principle of screw, i. e., small distance converts into large by measuring the orientation of the screw.

When measuring, the object is placed between the anvil and the spindle. Using the ratchet, the screw is moved until it touches the measured object, and the ratchet actuates (producing the controllable pressure on the measured objects).

A reading is again determined as a sum of two values – the reading in terms of the scale division values c_0 taken on the main scale and the reading in terms of the micrometer scale division values c taken on the rotating scale. Usually, the scale division value of the main scale is $c_0 = 0.5$ mm, and the rotation scale is divided into $N = 50$ intervals, resulting in the micrometer least count value $c = \frac{0.5}{50} = 0.01$ mm. *The instrument error for micrometer is determined in the same way as for the caliper: half of the least count value:*

$$\Delta x_a = \frac{c}{2} = \frac{0.01}{2} = 0.005 \text{ mm.}$$

The way to take a reading with micrometer is illustrated in Fig 6.8. The whole reading is taken as a sum of readings on the main scale and rotation scale. The main scale marks every mm (top panel) and a half of mm (bottom panel), and the reading on the main scale is given by its intersection with the edge of the thimble. The rotation scale is divided into 50 intervals, resulting to readings with the accuracy of 0.01 mm. To read the rotation scale, find its mark aligned with the datum line and multiply its value by the least count value.

How to use the micrometer:

1. Take a micrometer and check the zero error.
2. Place the object in between the anvil and spindle and rotate the thimble until the ratchet clicks once or twice.
3. Lock the lock nut to prevent further uncontrollable rotation of the spindle.
4. Take readings on the main scale and on the rotating scale.
5. Add both values: *Actual reading = Main scale + Rotating scale.*

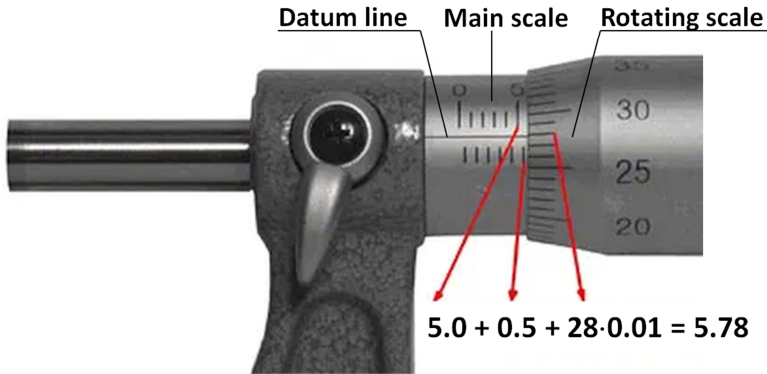


Fig. 6.8. Taking readings with a micrometer

6.2. Processing of measurement results

First of all, we should process the results of direct measurements.

Random errors are equally probable in sign. This means that when you carry out a sufficiently large number of experiments, approximately half of the experiments will have positive errors, and the other half of the experiments will have negative errors. When calculating the mean value, positive and negative errors are mutually compensated, and we get a more accurate result. Therefore *the best measurement value is the mean value* calculated as the arithmetic mean of all measurement results.

For example, when measuring the value of x , the mean value is calculated by the formula:

$$\langle x \rangle = \frac{1}{n} \sum_{i=1}^n x_i. \quad (6.5)$$

To determine the total (combined) error of the measurements, we first calculate the mean random error using the following formula:

$$\Delta x_r = \frac{1}{n} \sum_{i=1}^n |x_i - \langle x \rangle| = \frac{1}{n} \sum_{i=1}^n |\Delta x_r^i|. \quad (6.6)$$

The difference between the result of each measurement x_i and the best value $\langle x \rangle$ taken modulo so that the calculated random errors Δx_r^i

are not compensated when summing. After that, we calculate the *total error* Δx :

$$\Delta x = \sqrt{(\Delta x_r)^2 + (\Delta x_a)^2}. \quad (6.7)$$

The final result of the experiment will be represented in the form:

$$x = \langle x \rangle \pm \Delta x. \quad (6.8)$$

After processing the results of measurements of the cylinder dimensions, we obtain the best values of the diameter $\langle d \rangle$ and height $\langle h \rangle$, as well as the total errors Δd and Δh .

The best value of the mass of the cylinder is determined by substituting in the formula (6.1) the best values of the diameter and height:

$$\langle m \rangle = \rho \pi \frac{\langle d \rangle^2}{4} \langle h \rangle. \quad (6.9)$$

To determine *the error of indirect measurements of mass*, it is necessary to differentiate the working formula (Eq. 6.1):

$$\begin{aligned} m &= \left| \frac{\partial m}{\partial h} \right| \Delta h + \left| \frac{\partial m}{\partial d} \right| \Delta d + \left| \frac{\partial m}{\partial \rho} \right| \Delta \rho = \\ &= \rho \pi \frac{d^2}{4} \Delta h + \frac{2d\rho\pi h}{4} \Delta d + \pi \frac{d^2}{4} h \Delta \rho. \end{aligned} \quad (6.10)$$

To simplify the calculations, it is more convenient to start with calculating the total *relative mass error*:

$$\begin{aligned} \varepsilon_m = \frac{\Delta m}{m} &= \frac{1}{\rho \pi \frac{d^2}{4} h} \left(\rho \pi \frac{d^2}{4} \Delta h + \frac{2d\rho\pi h}{4} \Delta d + \pi \frac{d^2}{4} h \Delta \rho \right) = \\ &= \frac{\Delta h}{h} + \frac{2\Delta d}{d} + \frac{\Delta \rho}{\rho}; \end{aligned} \quad (6.11)$$

or

$$\varepsilon_m = \varepsilon_h + 2\varepsilon_d + \varepsilon_\rho. \quad (6.12)$$

Here $\varepsilon_d = \frac{\Delta d}{\langle d \rangle}$, $\varepsilon_h = \frac{\Delta h}{\langle h \rangle}$, and $\varepsilon_\rho = \frac{\Delta \rho}{\langle \rho \rangle}$ are relative errors of height, diameter and density, respectively. If the table from which the density of wood $\rho = 650 \text{ kg/m}^3$ was taken does not indicate the error $\Delta \rho$, then we assume $\Delta \rho \approx 0.5 \text{ kg/m}^3$.

Substituting the obtained values Δd and Δh and the known value of $\Delta \rho$ into Eq. 6.11, we obtain the value of the relative error ε_m .

Then, using the obtained value ε_m and the value $\langle m \rangle$ calculated with the Eq. 6.9, we calculate the *total absolute mass error* Δm :

$$\Delta m = \langle m \rangle \varepsilon_m. \quad (6.13)$$

The final result of the experiment should be presented in the form:

$$m = \langle m \rangle \pm \Delta m \text{ units; } \varepsilon_m = \dots\%. \quad (6.14)$$

6.3. Experiments

Experiment 1. Ruler measurements:

1.1. Calculate the instrument error of the ruler. Enter the data in the Table 6.1.

1.2. Measure the diameter d and the height h of a wood cylinder with a ruler several times (3–5 times). The diameter should be measured in different directions, and the height should be measured in different places of the cylinder. Fill Table 6.1 with the readings.

1.3. Calculate the mean values of $\langle d \rangle$ and $\langle h \rangle$, write the result in the table, *leaving one digit more*. Using the mean values, calculate the mean random errors Δd_r and Δh_r . Enter the data in the Table 6.1.

1.4. Evaluate the total error of the diameter Δd and the total error of the height Δh (see Eq. 6.7). Enter the data in the Table 6.1.

Table 6.1

Ruler measurements

n	Diameter $d, 10^{-3} \text{ m}$	Random error $\Delta d_r^i, 10^{-3} \text{ m}$	Height $h, 10^{-3} \text{ m}$	Random error $\Delta h_r^i, 10^{-3} \text{ m}$	Instrument error $\Delta_a, 10^{-3} \text{ m}$
1					
2					
3					
4					
5					
Mean values	$\langle d \rangle =$	$\langle d \rangle =$	$\langle h \rangle =$	$\Delta h_r =$	$\Delta_a =$
Total errors	$\Delta d =$	$\Delta h =$			

1.5. Write the final results of direct measurements of the diameter and the height in the form¹:

$$d = \langle d \rangle \pm \Delta d \text{ m}; \quad \varepsilon_d = \dots\%$$

$$h = \langle h \rangle \pm \Delta h \text{ m}; \quad \varepsilon_h = \dots\%$$

1.6. Calculate the mass of the cylinder (Eq. 6.9) using the mean values of $\langle d \rangle$ and $\langle h \rangle$ obtained with a ruler.

1.7. Evaluate the relative error ε_m (Eq. 6.11) and absolute error Δm (Eq. 6.13) of indirect measurement of the cylinder mass with a ruler.

1.8. Write the final result of indirect measurement of mass with a ruler in the form:

$$m^{\text{ruler}} = \langle m \rangle \pm \Delta m \text{ g}; \quad \varepsilon_m = \dots\%$$

¹ In error we leave one significant digit, always rounding up. The number of decimal places in the average value should be the same as in the error.

Experiment 2: Vernier caliper measurements:

2.1. Calculate the instrument error of the vernier caliper. Enter the data in the Table 6.2.

2.2. Measure the diameter d and the height h of a wood cylinder with a vernier caliper several times (3–5 times). The diameter should be measured in different directions, and the height should be measured in different places of the cylinder. Fill Table 6.2 with the readings.

2.3. Calculate the mean values of $\langle d \rangle$ and $\langle h \rangle$, write the result in the table, *leaving one digit more*. Using the mean values, calculate the mean random errors Δd_r and Δh_r in the Table 6.2.

2.4. Evaluate the total error of the diameter Δd and the total error of the height Δh (see Eq. 6.7). Enter the data in the Table 6.2.

Table 6.2

Vernier caliper measurements

n	Diameter $d, 10^{-3} \text{ m}$	Random error $\Delta d_r^i, 10^{-3} \text{ m}$	Height $h, 10^{-3} \text{ m}$	Random error $\Delta h_r^i, 10^{-3} \text{ m}$	Instrument error $\Delta_a, 10^{-3} \text{ m}$
1					
2					
3					
4					
5					
Mean values	$\langle d \rangle =$	$\langle d \rangle =$	$\langle h \rangle =$	$\Delta h_r =$	$\Delta_a =$
Total errors	$\Delta d =$	$\Delta h =$			

2.5. Write the final results of direct measurements of the diameter and the height in the form:

$$d = \langle d \rangle \pm \Delta d \text{ m}; \quad \varepsilon_d = \dots\%$$

$$h = \langle h \rangle \pm \Delta h \text{ m}; \quad \varepsilon_h = \dots\%$$

2.6. Calculate the mass of the cylinder (Eq. 6.9) using the mean values of $\langle d \rangle$ and $\langle h \rangle$ obtained with a vernier caliper.

2.7. Evaluate the relative error ε_m (Eq. 6.11) and absolute error Δm (Eq. 6.13) of indirect measurement of the cylinder mass with a vernier caliper.

2.8. Write the final result of indirect measurement of mass with a vernier caliper in the form:

$$m^{\text{caliper}} = \langle m \rangle \pm \Delta m \text{ g}; \quad \varepsilon_m = \dots\%$$

Experiment 3: Micrometer measurements:

3.1. Calculate the instrument error of the micrometer. Enter the data in the Table 6.3.

Table 6.3

Micrometer measurements

n	Diameter $d, 10^{-3} \text{ m}$	Random error $\Delta d_r^i, 10^{-3} \text{ m}$	Height $h, 10^{-3} \text{ m}$	Random error $\Delta h_r^i, 10^{-3} \text{ m}$	Instrument error $\Delta_a, 10^{-3} \text{ m}$
1					
2					
3					
4					
5					
Mean values	$\langle d \rangle =$	$\langle d \rangle =$	$\langle h \rangle =$	$\Delta h_r =$	$\Delta_a =$
Total errors	$\Delta d =$	$\Delta h =$			

3.2. Measure the diameter d and the height h of a wood cylinder with a micrometer several times (3–5 times). The diameter should be measured in different directions, and the height should be measured in different places of the cylinder. Fill Table 6.3 with the readings.

3.3. Calculate the mean values of $\langle d \rangle$ and $\langle h \rangle$, write the result in the table, *leaving one digit more*. Using the mean values, calculate the mean random errors Δd_r and Δh_r . Enter the data in the Table 6.3.

3.4. Evaluate the total error of the diameter Δd and the total error of the height Δh (see Eq. 6.7). Enter the data in the Table 6.3.

3.5. Write the final results of direct measurements of the diameter and the height in the form:

$$d = \langle d \rangle \pm \Delta d \text{ m}; \quad \varepsilon_d = \dots\%$$

$$h = \langle h \rangle \pm \Delta h \text{ m}; \quad \varepsilon_h = \dots\%$$

3.6. Calculate the mass of the cylinder (Eq. 6.9) using the mean values of $\langle d \rangle$ and $\langle h \rangle$ obtained with a micrometer.

3.7. Evaluate the relative error ε_m (Eq. 6.11) and absolute error Δm (Eq. 6.13) of indirect measurement of the cylinder mass with a micrometer.

3.8. Write the final result of indirect measurement of mass with a micrometer in the form:

$$m^{\text{micro}} = \langle m \rangle \pm \Delta m \text{ g}; \quad \varepsilon_m = \dots\%$$

Discussion

1. Compare the results of mass measurement with three instruments. Are they in agreement? If not, explain the cause of the discrepancy.

2. Which instrument allowed getting the most accurate result? What is the basis for such a conclusion?

3. Measurements of which quantity gave the greatest contribution to the mass error and why?

4. Make the report.

5. Answer the control questions.

7. CONTROL QUESTIONS

1. What is a measurement? What types of measurements do you know?
2. What is the meaning of confidence probability?
3. What are absolute and relative measurement errors? How are they related? How is absolute error related to confidence interval?
4. What is measurement accuracy? How does it relate to confidence interval?
5. In what case the results of measurements of the same physical quantity by different instruments are in agreement?
6. How are measurement errors classified?
7. What is a random measurement error? Where does it come from? Is it possible to eliminate it?
8. What is the systematic measurement error? Where does it come from? Is it possible to eliminate it?
9. What is the instrument measurement error? Where does it come from? Is it possible to eliminate it?
10. What is a gross error? Where does it come from?
11. What types of errors should be taken into account when performing laboratory work in a physical workshop?
12. What value is taken as the best value of the measured value? How to determine it?
13. Write the formulae to calculate random error.
14. What is the accuracy class? How to determine the instrument error with and without information about the accuracy class?
15. How to calculate total measurement error?
16. Describe the algorithms for processing direct measurements.
17. What formulae are used to calculate the error of indirect measurements? Specify in which cases they can be applied.
18. Describe the algorithm for processing indirect measurements.
19. What is the least count of the vernier scale? How can we estimate its value for the vernier caliper and micrometer?
20. How the instrument error of devices with a vernier scale (caliper, micrometer) is estimated?
21. How to take caliper readings?
22. How to take micrometer readings?

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Appendix A

Powers of ten and SI prefixes

Power	Prefix	Abbreviation
10^{18}	exa	E
10^{15}	peta	P
10^{12}	tera	T
10^9	giga	G
10^6	mega	M
10^3	kilo	k
10^2	hecto	h
10^1	deka	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

Appendix B

Greek alphabet

Alpha	A	α
Beta	B	β
Gamma	Γ	γ
Delta	Δ	δ
Epsilon	E	ϵ
Zeta	Z	ζ
Eta	H	η
Theta	Θ	θ
Iota	I	ι

End of appendix B

Kappa	K	κ
Lambda	Λ	λ
Mu	M	μ
Nu	N	ν
Xi	Ξ	ξ
Omicron	O	\omicron
Pi	Π	π
Rho	P	ρ
Sigma	Σ	σ
Tau	T	τ
Upsilon	Υ	υ
Phi	Φ	ϕ, φ
Chi	X	χ
Psi	Ψ	ψ
Omega	Ω	ω

Appendix C

Student's coefficients $t_{p,n}$

Number of measurements n	Confidence probability p		
	0.9	0.95	0.99
2	6.31	12.71	63.66
3	2.92	4.30	9.92
4	2.35	3.18	5.84
5	2.13	2.78	4.60
6	2.02	2.57	4.03
7	1.94	2.45	3.71
8	1.90	2.37	3.50
10	1.83	2.26	3.25
20	1.73	2.09	2.86
Infinite	1.65	1.96	2.58

Rules for calculations with approximate numbers

Significant digits in an approximate number

All the digits in the number are significant digits with one exception: if the digit is a zero that is used just to locate the decimal point then it is not significant (fig. D.1).

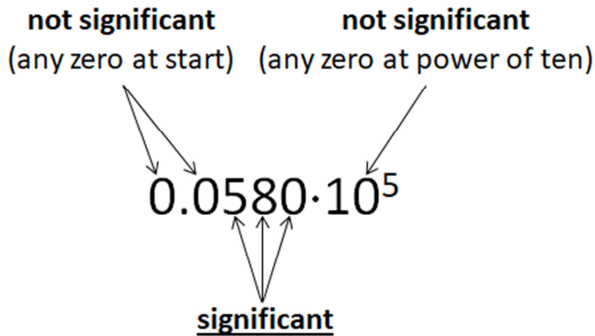


Fig. D.1. To definition of significant digits: three significant digits are detected

Rounding when calculating approximate numbers:

1. When *adding and subtracting* approximate numbers, as a *final result*, one should leave as many decimal places as there were in the number with the least number of decimal places:

$$4.22 + 1.6 = 5.82 \approx 5.8;$$

$$3.055 - 1.05 + 8.6 = 10.605 \approx 10.6;$$

$$7 + 20.6 = 27.6 \approx 28.$$

2. When *multiplying and dividing*, as a *final result*, one should leave as many significant digits as there were in the number with the least number of significant digits:

$$5.20 \cdot 3.0 = 15.60 \approx 16;$$

$$\frac{3 \cdot 10^5}{1.5 \cdot 10^3} \cdot 4.00 \cdot 10^2 \approx 8 \cdot 10^4;$$

$$\frac{6.25 \cdot 20}{0.5} = 250 \approx 3 \cdot 10^2.$$

3. When *raising an approximate number to a power (extracting a root)*, the final result must have as many significant figures as there are in the base (radical expression):

$$5.20^3 = 140.608 \approx 141;$$

$$\sqrt{9.00 \cdot 10^6} \approx 3.00 \cdot 10^3;$$

$$\sqrt{500} = 22.3607 \approx 22.4.$$

4. When *taking a logarithm*, the mantissa stores as many significant digits as there are in the original number:

$$\ln 5.20 = 1.6487 \approx 1.65;$$

$$\lg 250 = 2.39794 \approx 2.40;$$

$$\text{if } \ln x = 1.80, \text{ then } x = \exp(1.80) = 6.049647 \approx 6.05.$$

5. When recording *the results of intermediate calculations*, one should save a reserve digit (the digit to the right of the doubtful one):

$$4.22 \cdot 1.6 = 6.752 \approx 6.75.$$

For the intermediate result keep the reserve digit 5 after the doubtful digit 7. For the final result, the correct rounding would be 6.8:

$$\frac{6.444 \cdot 10^3}{2.0} = 3.222 \cdot 10^3 \approx 3.22 \cdot 10^3.$$

6. If *tabular data* is used in the calculations, then all of their numbers are correct.

Expressions of absolute and relative errors for indirect measurements

Function (Indirect Measurement)	Absolute Error ΔF	Relative Error $\varepsilon_F = \frac{\Delta F}{F}$
$F = \alpha x$	$\Delta F = \alpha \Delta x$	$\varepsilon_F = \frac{\Delta x}{x} = \varepsilon_x$
$F = \alpha x^n$	$\Delta F = \alpha n x^{n-1} \Delta x = \left \frac{nF}{x} \right \Delta x$	$\varepsilon_F = n \varepsilon_x$
$F = x + y$	$\Delta F = \Delta x + \Delta y$	$\varepsilon_F = \frac{\Delta x + \Delta y}{ x + y }$
$F = x - y$	$\Delta F = \Delta x + \Delta y$	$\varepsilon_F = \frac{\Delta x + \Delta y}{ x - y }$
$F = xy$	$\Delta F = y \Delta x + x \Delta y$	$\varepsilon_F = \left \frac{\Delta x}{x} \right + \left \frac{\Delta y}{y} \right = \varepsilon_x + \varepsilon_y$
$F = \frac{x}{y}$	$\Delta F = \left \frac{1}{y} \right \Delta x + \left \frac{x}{y^2} \right \Delta y$	$\varepsilon_F = \left \frac{\Delta x}{x} \right + \left \frac{\Delta y}{y} \right = \varepsilon_x + \varepsilon_y$
$F = \alpha e^{\beta x}$	$\Delta F = \alpha \beta e^{\beta x} \Delta x = \beta F \Delta x$	$\varepsilon_F = \beta \Delta x$
$F = \alpha \ln(\beta x)$	$\Delta F = \left \frac{\alpha}{x} \right \Delta x = \alpha \varepsilon_x$	$\varepsilon_F = \left \frac{\alpha}{F} \right \varepsilon_x$
$F = \sin \alpha x$	$\Delta F = \alpha \cos \alpha x \Delta x$	$\varepsilon_F = \alpha \operatorname{ctg} \alpha x \Delta x$
$F = \cos \alpha x$	$\Delta F = \alpha \sin \alpha x \Delta x$	$\varepsilon_F = \alpha \operatorname{tg} \alpha x \Delta x$

Footnote: Here α , β and n are constants, x and y are variables.

Учебное издание

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**ПОГРЕШНОСТИ ИЗМЕРЕНИЙ.
АНАЛИЗ И ОЦЕНКА**

Пособие

для студентов специальности

1-53 01 01 «Автоматизация технологических процессов
и производств (по направлениям)»

На английском языке

Редактор *Е. И. Беницевич*

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