Ministry of Education of the Republic of Belarus

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STRUCTURAL MECHANICS

Approved by the Ministry of Education
of the Republic of Belarus as a textbook
for students of higher education institutions
for construction specialties

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The textbook reveals the basic principles of calculating statically determinate and statically indeterminate systems for static loads and effects. Examples of calculation are given.

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FOREWORD

This textbook is intended for students of construction specialties of higher educational institutions and universities. In a relatively small volume of the present work with sufficient completeness and degree of accuracy, theoretical information is presented and detailed solutions to practical problems in structural mechanics are given.

The textbook is written on the basis of long term extensive experience during teaching a course in structural mechanics at the Belarusian National Technical University. The specific gravity of individual chapters is slightly increased in comparison with their traditional contents, since the material presented in them can be useful for specialists working in the field of structural mechanics applications for the development of both construction design and computer software.

The authors believe that the successful solution of applied problems by students will be possible if they study the theoretical material in depth.

Additional educational literature on the course of structural mechanics is presented at the end of the book.

Comments on possible shortcomings of the textbook will be greatly appreciated by the authors.

THEME 1. GENERAL CONVENTIONS AND CONCEPTS OF STRUCTURAL MECHANICS

1.1. Tasks and Methods of Structural Mechanics

Structural mechanics as a science develops the theory of creating engineering structures and methods for calculating their strength, rigidity and stability under a variety of static and dynamic loads and other influences. Strength analysis involves the determination of internal forces in all elements of a structure. Based on the found internal forces, the strength and stability of each element of the structure, as well as the strength and stability of the entire structure as a whole, are checked. The rigidity of the structure is estimated by determining the displacements (linear and angular) of its characteristic points, sections, elements and comparing the found displacements values with the normalized values.

In the training curriculum for civil engineers, structural mechanics immediately follows such a discipline as resistance of materials. The resistance of materials studies the behavior under load of individual elements: bars, beams, columns, plates. The structural mechanics study the response of entire complex structures composed of bars, plates, and solids, as well as connecting and supporting devices (nodes, links, constraints, etc.).

The main tasks of structural mechanics are:

• Study of the laws of structures formation.

• Development of methods for analyzing the internal forces in the elements and parts of structures due to various external influences and loads.

• Development of methods for determining displacements and deformations.

• Study of stability conditions of structures equilibrium in a deformed state.

• Study of the structures interaction with the environment.

• Study of changes in the stress-strain state of structures during their long-term operation.

In practical terms, the so-called direct task of structural mechanics is most fully developed: determination of the stress-strain state of a structure under given loads and other influences. It is assumed that the design scheme of the structure, the properties of the materials and the dimensions of its elements are also given. This main task of structural mechanics is sometimes called the verification calculation of the structure.

In the calculations of buildings and engineering structures, the hypothesis of continuity of materials, the hypothesis of their homogeneity and isotropy, the hypothesis of direct proportionality between stresses and strains are used. Deformations and displacements of structural elements are assumed to be small, which allows the analysis of most structures using an undeformed design scheme.

To solve the problems, structural mechanics develops and applies theoretical and experimental methods. Theoretical methods use the achievements of theoretical mechanics, higher and computational mathematics, computer science and programming. Experimental methods are based on testing samples, models and real structures.

Though at the initial stage of its development structural mechanics was based mainly on graphical methods for solving its problems, then with the development of computer technology analytical solutions have become more and more applied. Moreover, instead of numerous particular methods and techniques that made it possible to avoid solving systems of joint equations, nowadays in structural mechanics, general universal methods (analytical and numerical) have come to the fore, allowing engineers to analyze complex structures as entire deformable systems. The solution of systems of joint linear algebraic equations with hundreds of thousands of unknowns has ceased to be a stumbling block. Computer technology has allowed not only to solve, but also to compose systems of equations of high orders, and most importantly, to review the obtained results, displaying them on the monitor screen in a graphical form familiar to an engineer.

Structural mechanics is a constantly developing applied science. New mathematical models of the real materials behavior during their deformation are being developed. The loading conditions of structures and the values of loads are being specified. Thermal and other effects are being taken into account. Nonlinear methods for analyzing structures in a deformed state are increasingly being used. Methods of synthesis and design optimization of structures are being developed. The connection of structural mechanics with the design of structures, with the technology of their manufacture and construction, is becoming increasingly close. It all leads to the creation of more solid, economical, reliable and durable buildings and structures.

1.2. Design Scheme of the Structure. The Concept and Elements

When analyzing structures, engineers usually do not deal with the real structure itself, but with its design scheme. The choice of a design scheme is a very important and responsible process. The design scheme should reflect the actual response of the structure, as close as possible, and, if possible, facilitate both the calculation process itself and the analysis process of the calculation results. In this respect it is essential to have extensive experience in the calculation of structures, to have a good idea of the analyzed structure behavior. It is necessary to know and to be able to predict the impact of the individual elements on the response of the entire construction.

Depending on the geometric dimensions in structural mechanics, the following main structural elements are distinguished: rods or bars, shells, plates, solids, thin-walled bars. Structural elements may also include connecting devices (nodes, links, and other connections) and supporting or limiting devices (supports or constraints).

Spatial structural elements, in which one size (length) significantly exceeds the other two, are called bar elements.

Spatial elements, one size (thickness) of which is much smaller than the other two sizes, are called shells, if they are bounded by two curved surfaces or plates if they are bounded by two planes.

On the design schemes of the structures, the bars are replaced by their axial lines (straight line, curve line or polyline), and the plates and shells are replaced by their median surfaces (plane or curved).

Solid bodies are elements of the structure or the environment in which all three sizes are of the same order (sometimes unlimited), for example: foundations, dams, retaining walls, and soil and rock massifs.

Bars are called thin-walled if they have all main dimensions of different orders: the thickness is significantly less than the cross-sectional dimensions, and the dimensions of the cross-section are much smaller than the length.

Separate elements that form the structure are combined into a united system through nodal connections, or simply nodes. Nodes are also considered as idealized. Usually they are divided into nodes that connect the elements by ideal hinges without friction and the nodes that are absolutely rigid.

An ideal hinged node (or simply a hinge) is considered as a device that allows only mutual rotation of the connected elements relative to each other. At the design schemes, the hinge is indicated by a small circle.

Hinged joint transfers only concentrated force from element to element. This force is usually decomposed into two components. When two rectilinear elements lying on one straight line are articulated by hinge (Figure 1.1, a), the internal force in the joint is decomposed into longitudinal N and transverse Q components. When the elements are articulated at an angle (Figure 1.1, b), the interaction force is decomposed into vertical V and horizontal H components, or otherwise. There is no bending moment in any swivel joint (in any hinge).

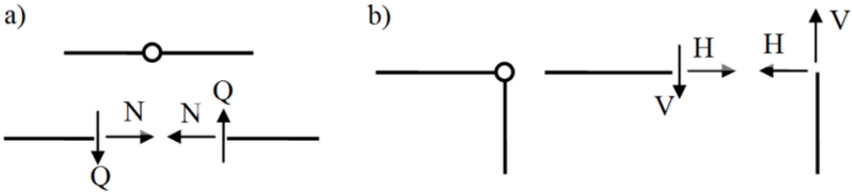


Figure 1.1

An absolutely rigid connection of elements (rigid node) completely eliminates all their mutual displacements. Special designations for rigid nodes are not usually introduced (Figure 1.2, a). Sometimes a rigid node is designated as a small square (Figure 1.2, b). Three internal forces act in a rigid node, for example, the vertical component V, the horizontal component H and the bending moment M (Figure 1.2, c).

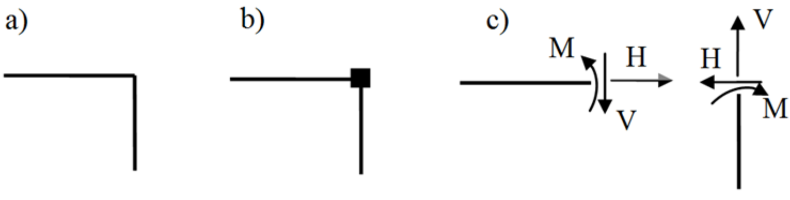


Figure 1.2

Sometimes such division of nodes into perfectly hinged and ideally rigid is not true. Then the nodes are considered as compliant or elastic, allowing mutual displacements of the connected elements (for example, rotation) proportional to the internal forces acting in the node. On design schemes, elastic nodes are being depicted with additional elements: deformable (Figure 1.3, a) and/or absolutely rigid (Figure 1.3, b) and others. Internal forces in elastic nodes depend on the mutual displacement of the connected elements. For example, the value of the bending moment (Figure 1.3, c) in an elastic node (Figure 1.3, a, b) depends on the mutual rotation angle of the connected bars.

A structure is attached to the ground (to the foundation) or to other structures with the help of supports. There are the following main types of disign schemes for supports of plane (two-dimensional) structures: hinged movable supports (roller supports), hinged immovable supports (pin supports), absolutely rigid supports (build-in or fixed supports), movable rigid supports and floating rigid supports. The latter eliminates only rotation.

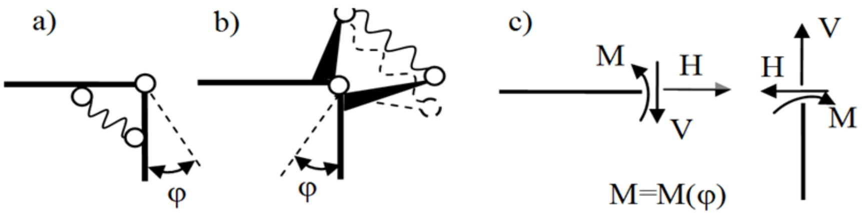


Figure 1.3

The hinged movable support limits only one linear movement in a given direction. Structurally, such a support can be made in the form of a cylindrical roller. The roller is freely moving along the supporting surface (Figure 1.4, a). A single reactive force arises in such support. The action line of the reactive force passes through the points of contact of the roller with the supporting surfaces of the foundation and structure. If the displacements of the real structure are small enough, then the roller can be replaced with a swinging rod (Figure 1.4, b, c). In the design schemes, the hinged movable support is depicted in the form of one rectilinear support rod with hinges at the ends (Figure 1.4, c). In such support, the direction of the reactive force coincides with the direction of the support rod, i.e. with the direction of the prohibited displacement.

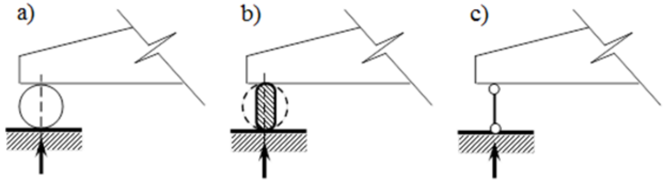


Figure 1.4

If large displacements of the support point are possible in the structure, then the design diagram of the articulated movable support is depicted in the form of a slide-block pivotally connected to the structure and freely sliding on the supporting surface (Figure 1.5, a), or freely rolling on it on rollers (Figure 1.5, b). The structure cannot move in the direction perpendicular to the supporting surface. A single reactive force normal to the supporting surface acts on the structure from the side of such roller support.

Even if the reaction of a roller support, which is depicted in the form of an inclined support rod, is decomposed into two components (Figure 1.5, c), then only one of them will be unknown. The second is clearly expressed through the first.

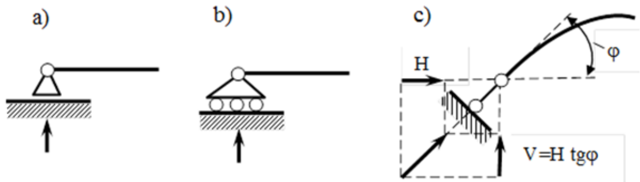


Figure 1.5

An immovable hinged support (Figure 1.6) completely eliminates all linear displacements and allows free rotation only about the axis of the support hinge. In this support, only a reactive force arises, the action line of which passes through the center (axis) of the pinned support. Since the direction of the action line of this reaction is unknown, to define this reaction it is decomposed into two unknown components, usually vertical and horizontal. Therefore, it is possible to assume that the hinged immovable support (Figures 1.6, a, b) is equivalent to two support rods intersecting on the axis of the support hinge (Figures 1.6, c, d).

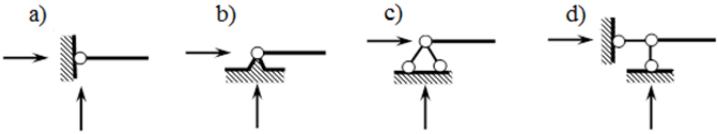


Figure 1.6

An absolutely rigid support (Figure 1.7, a), does not allow either linear or angular movements. Three reactions arise in such fixed support: two reactive forces (two components of the total reactive force of an unknown direction) and a reactive moment. The absolutely rigid support is equivalent to three support rods (Figure 1.7, b).

Rigid movable (non-hinged) supports leave freedom for one linear displacement (Figure 1.7, c, e). Naturally, the reactive force component in the rigid movable supports in the direction of free linear movement is absent. There a reactive moment remains and a reactive force perpendicular to the free linear displacement remains, i.e., two support reactions. Such rigid movable supports are equivalent to two support rods (Figures 1.7, d, f).

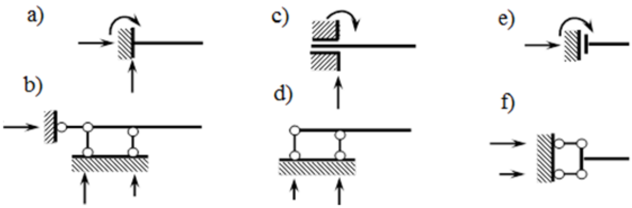


Figure 1.7

Floating rigid supports (Figure 1.8) eliminate only angular displacements. Only one reactive moment arises in a floating support. Floating support can be designated by a special device (Figure 1.8, a), or simply by a square (Figure 1.8, b), specifying its properties.

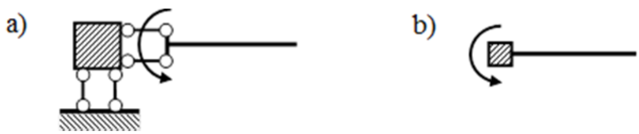


Figure 1.8

Modern methods of structural mechanics, modern computer technology and modern design and computing systems for the analysis of structures allow you to calculate almost any design scheme.

For the same framework, it is possible to choose several design schemes. Preliminary design of cross section parameters of structural elements can be performed on a calculator according to a simplified calculation scheme. The final calculation should be performed in accordance with more complex and accurate design schemes using computers and modern software.

Here is an example of choosing a design scheme for truss structure. Under certain conditions, a system of rods with ideal frictionless hinged joints on each end can be adopted as a design scheme for its analysis (Figure 1.9). In this case the analysis of internal forces in its elements is easily performed on a calculator with the use of equilibrium equations only.

In fact truss structures can be made of bent-welded rectangular or tube profiles with welded nodes or in monolithic reinforced concrete, then their analysis will require a more accurate design scheme with rigid nodal joints (Figure 1.10).

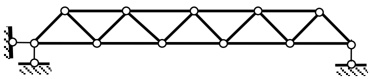


Figure 1.9

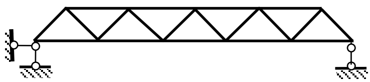


Figure 1.10

Such design scheme is already statically indeterminate many times. Its analysis is possible when taking into account additional deformation equations and is reduced to solving a system of joint linear algebraic equations of a sufficiently high order. It will require the use of computer software.

1.3. Classification of Design Schemes of Structures

Classification of structures can be performed, in terms of their analysis, according to various criteria.

1.3.1. Plane and Spatial Structures

A structure is called plane, or two-dimensional, if:

a) the geometric axis of all its elements that make up the structure lie in the same plane,

b) in all cross sections of each element one of the main axes of inertia lies in the same plane,

c) the lines of action of all the loads applied to the structure also lie in the same plane.

If at least one of these conditions is not fulfilled, then the structure is spatial.

All real structures are spatial. But in order to simplify their analysis, they are divided into a number of plane systems. Such dismemberment is not always possible. Therefore, some structures have to be considered as spatial. This book is devoted to the analysis and calculation of predominantly plane systems.

1.3.2. Bars Systems, Thin-walled Spatial Systems and Massifs Systems (Solid Bodies).

Structures which are consisted of rectilinear or curvilinear bars or rods are called bar systems.

Structures composed of shells and plates are called thin-walled and are usually spatial.

Massifs systems mean structures consisting of solid bodies, for example: foundations, dams, retaining walls, as well as soil and rock massifs themselves. Massive systems can be considered both in three-dimensional and in two-dimensional space.

Traditionally, structural mechanics deals with the study of mainly bar systems. But modern computer software allows you to analyze spatial thin-walled and massifs systems, using almost the same methods as for bar systems.

1.3.3. Structures with Hinged or Rigid Nodal Connections of Elements

A bar system composed of rods with ideal frictionless hinge joints only on each end of each rod is called a hinge-rod system or a truss (Figure 1.9).The bar system, in which the elements are connected, basically absolutely rigidly, is called a frame (Figure 1.10, Figure 1.11).

In the same structure, both hinged and rigid joints of elements can be used. Sometimes this method of joining is called combined. As example it is the beam with a polygonal complex tie (Figure 1.12). The simultaneous use of rigid and articulated joints takes place in the design schemes of many other types of structures, for example: in a three-hinged frame (Figure 1.13), in a two-span two-tier frame with a central pendulum column and with a pivotally supported upper crossbar (Figure 1.14).

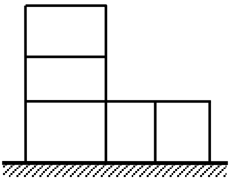


Figure 1.11

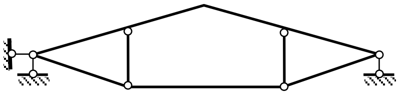
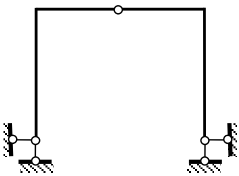


Figure 1.12



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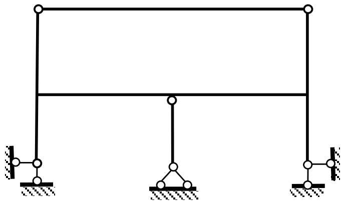


Figure 1.14

1.3.4. Geometrically Changeable and Unchangeable Systems. Instantaneously Changeable and Instantaneously Rigid Systems

If a structural system allows a change in its geometry (shape distortion) due to the mutual displacement of the elements without their deformation or destruction, then the following system is called geometrically changeable (Figure 1.15). If a change in the shape (geometry) of a system is possible only due to deformation or destruction of its elements, then the following system is geometrically unchangeable (Figure 1.16).

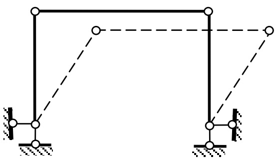


Figure 1.15

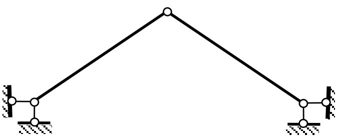


Figure 1.16

The classification of structures by kinematic characteristics is of great importance, since, as a rule, geometrically unchangeable systems can be used as structures. Only some hanging systems of a variable type made of flexible elements or cables are an exception.

With an arbitrary change in the sizes of the elements and/or a change in the mutual arrangement of the nodes of the system, it is possible to obtain its special (singular) shape, the kinematic properties of which will differ from the properties of adjacent forms. For example, a two-rod geometrically unchangeable system (Figure 1.16), when changing the lengths of its elements, can take a special form in which both rods will lie on one straight line (Figure 1.17).

In this special case, the intermediate joint will be free to move vertically. However, the vertical movement of the intermediate joint can only be infinitesimal, since the rods are assumed to be completely non-deformable, i.e. absolutely rigid. All adjacent forms in which the rods do not lie on one straight line will be geometrically unchangeable. Special forms in which the system allows infinitely small movements are called instantaneously changeable. When a system is removed from an instantaneously changeable configuration, it becomes geometrically unchangeable.

Systems whose configurations are instantaneously changeable (Figure 1.17) or close to those (Figure 1.18), as a rule, are not used as structures, since they have heightened deformability.

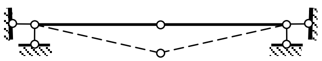


Figure 1.17

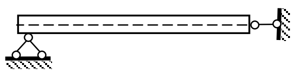


Figure 1.18

On the other hand, in a geometrically changeable system (Figure 1.19), one can choose the lengths of its elements so that, for example, all its nodes are located on one straight line (Figure 1.20). This will be a special form of a geometrically changeable system, which is called instantaneously rigid. In adjacent forms, the considered geometrically changeable system allows large kinematic movements without deformations of its elements (Figure 1.19). The same system in a special form (Figure 1.20) under the condition of absolute inextensibility of the rods allows only infinitesimal displacements.

Thus, both geometrically unchangeable and geometrically changeable systems can have special, singular forms.

Under real conditions, when elements of structures are made of deformable materials, singular forms are characterized by finite displacements of nodes, the values of which are an order of magnitude higher than the elongations of elements. Consequently, instantaneously changeable systems are characterized by heightened deformability compared to geometrically unchanged systems, and instantaneously rigid systems are characterized by heightened stiffness compared to geometrically changeable systems.

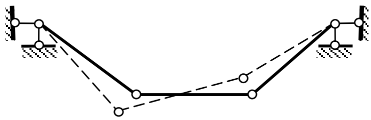


Figure 1.19

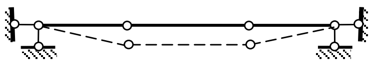


Figure 1.20

Instantaneously rigid systems are widely used in pre-stressed suspension and cable-stayed systems

1.3.5. Thrust and Free Thrust Systems

If in a structure a load of one direction causes support reactions of the same direction, then such a structure is called free of thrust or simply non-thrusting. All other structures can be attributed to thrusting systems. The thrust of a structure is support reactions normal to the load action direction.

A classic example of non-thrusting systems is beams: a simply supported rectilinear beam (Figure 1.21), a simply supported curvilinear beam (Figure 1.22) and other beam-type systems (Figures 1.9, 1.10, 1.12). The double-hinged arch (Figure 1.23) and the three-hinged frame (Figure 1.24), the same as many others, are thrusting systems.

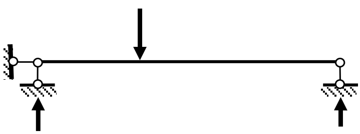
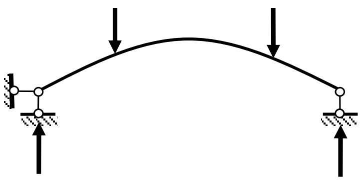


Figure 1.21



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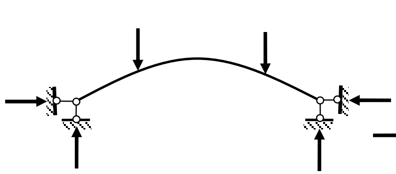


Figure 1.23

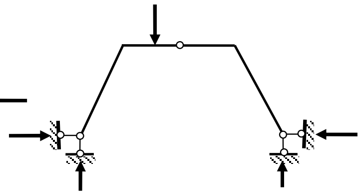


Figure 1.24

Therefore non-thrusting systems are often called as beam systems. And thrusting systems are called as arch systems.

1.3.6. Statically Determinate and Indeterminate Systems

In a statically determinate system, all internal forces can be found using only equilibrium equations (static equations).

If there is a need to use the equations of deformations to determine the support reactions or at least part of the internal forces, then such a system is called statically indeterminate

A statically indeterminate system has an excess of nodal and other connections or links than is necessary for its geometric immutability. A statically indeterminate system can have preliminary stress (initial internal forces, i.e., forces without load due to thermal effects, displacement of supports, inaccurate assembly, etc.). In a statically determinate system, initial internal forces are impossible without external loads.

1.3.7. Linearly and Nonlinearly Deformable Systems

If the relations between the load applied to the structure and the internal forces and displacements caused by it obeys the law of direct proportionality, then such a structure is called linearly deformable, or simply linear. In a linearly deformable system, deformations and displacements are supposed to be small. Their influence on the distribution of internal forces is neglected. The geometry of the deformed structure is assumed to coincide with the geometry of the original undeformed structure. The equilibrium equations are relative to the original, undeformed design scheme. The stress-strain state of a linear system is described by linear differential or linear algebraic equations.

However, if the deformations and displacements caused by external influences in a structure are significant, then the relations between the loads, the internal forces and displacements become non-linear. Such a structure is called nonlinearly deformable, or non-linear.

Non-linearity due to a change in the geometry of the design scheme of the structure is called geometric non-linearity. The calculation of large-span and high-rise structures is usually carried out taking into account geometric nonlinearity. All geometrically changeable, instantaneously changeable and instantaneously rigid systems (suspension coverings and roofs, suspension bridges, cable and cable-stayed networks and systems) are geometrically non-linear.

The nonlinearity associated with the deviation of the law of deformation of the building material from the law of direct proportionality, Hooke's law, is called physical nonlinearity.

1.4. Plane Bar System Degree of Freedom

The degree of freedom of a body or system of bodies is the number of independent geometric parameters that determine the position of a body or system of bodies when they move on a plane or in space.

The position on the plane of a movable (free) material point of infinitesimal dimensions (hinge node) is characterized by its two coordinates relative to an arbitrary fixed reference system located in the same plane (Figure 1.25). Consequently, the point (hinge node) has two degrees of freedom on the plane.

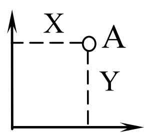


Figure 1.25

A separate body (bar) or a knowingly geometrically unchangeable system of bodies (bars system) or its part, which can move on a plane as a whole, without changing its geometric shape, is called a disk.

The position of the moving (free) plane body (disk) on the plane is characterized by three independent parameters, for example: the abscissa x and the ordinate y of a point A and the angle of some straight line AB belongs to the disk (Figure 1.26). Thus, when moving on a plane the disk has three degrees of freedom. A rigid node on a plane, even of sufficiently small dimensions, in contrast to the articulated node, should be considered as a disk. Therefore, a rigid node on a plane has three degrees of freedom.

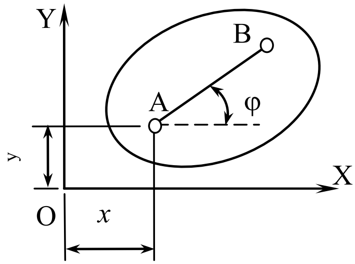


Figure 1.26

In space, a free solid is considered as a spatial block and has six degrees of freedom: three coordinates of any of its points and three angles of rotation of any of its lines with respect to the axes of the fixed spatial coordinate system.

In this section only plane systems are considered.

1.4.1. Classification of Plane Systems Connections

Any device that reduces the degree of freedom of a body or system of bodies by one is called a simple connection or a simple link or a single constraint. If the device constrains several degrees of freedom, then it is considered as a complex (multiple) connection, equivalent to several simple ones.

Each connection has both kinematic and static characteristics.

The kinematic characteristic determines the types of motion of one disk relative to another, which are constrained by the connection, the number of degrees of freedom that this connection eliminates. The static characteristic determines the number and types of reactions that occur in the corresponding connection.

Thus, any structure can be considered as a system of disks connected by links, both among themselves and with a supporting surface (ground).

The earth (supporting surface) can also be considered as a disk. Most often, an immovable coordinate system is associated with the ground, and the degree of freedom of the system under study is determined relative to the earth.

In kinematic analysis, disks and connections are assumed to be nondeformable, absolutely rigid.

Let's consider the design schemes of connections used in structural mechanics.

A movable hinged support is equivalent to one simple link. A disk, which is attached to the ground (supporting surface) with a movable hinged support, loses one degree of freedom. A system of a disk and a support rod has two degrees of freedom (Figure 1.27).

A single hinged rod connecting two disks can also be considered as a simple link. A system of two disks connected by one hinged rod loses one degree of freedom (Figure 1.28). The total degree of freedom of such a system is five, as opposed to six degrees of freedom for two free disks.

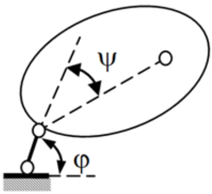


Figure 1.27

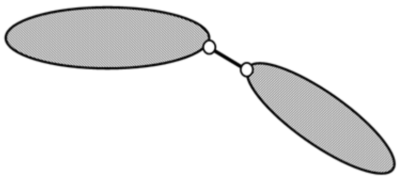


Figure 1.28

A single hinge (indicated by a circle on the design diagrams) is equivalent to two simple links. Connecting two disks, one hinge reduces their total degree of freedom, equal to six, to four. The position of two disks connected by the hinge is characterized by two coordinates x and y of point A and two angles φ and ψ fixing the position of lines AB and BC (Figure 1.29, a). The earth (supporting surface) can be considered as an immovable disk. A movable disk, when it is attached by a hinge to the ground (to a fixed supporting surface), loses two degrees of freedom. The position of this disk is characterized by only one angle of rotation relative to the axis of the hinge (Figure 1.29, b). Such a device can be considered as an immovable hinged support, equivalent to two simple support rods (Figure 1.29, c). An immovable hinged support eliminates two degrees of freedom.

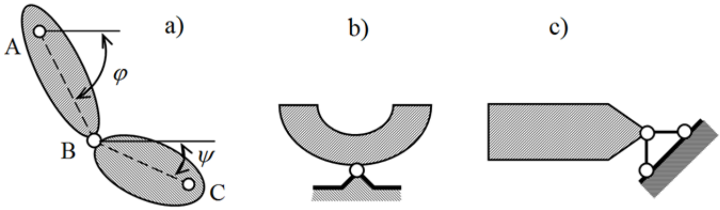


Figure 1.29

A system of three disks connected by two hinges (Figure 1.30, a) has five degrees of freedom. Two hinges eliminated four degrees of freedom. In this system, the intermediate disk can also be considered as a simple connection (compare with the system in Figure 1.28).

In kinematic analysis, any rod (bar) can be considered as a disk, and any disk can be replaced by a bar.

Often two hinges connecting three disks come together and merge, as if into one hinge on a common axis (Figure 1.30, b). Such a complex hinge is equivalent to two simple hinges, or four simple links.

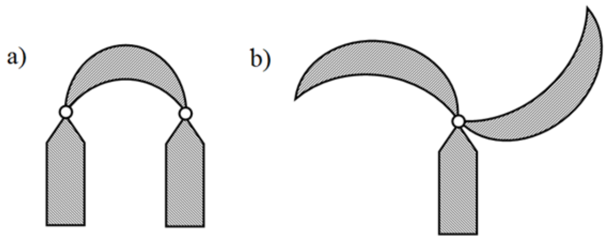


Figure 1.30

In the general case, the multiplicity of the following complex hinge is one point less than the number of disks (rods) connected on one axis. In other words, the relation is true:

$$H = D - 1,$$

where H is the multiplicity of the complex hinge, D is the number of disks connected by the complex hinge on one axis.

Examples of simple hinges are shown in Figure 1.31, a. Figure 1.31, b shows multiple hinges.

If two disks (rods) are monolithically (or by welding) combined into one disk, then such a joint is called a rigid connection, or a rigid node.

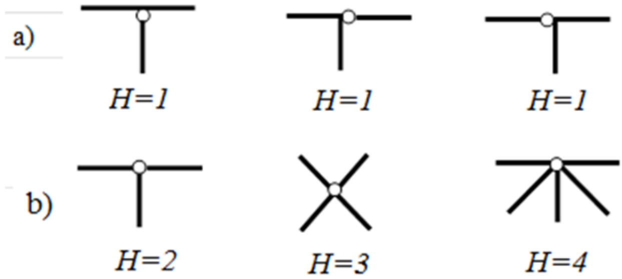


Figure 1.31

Rigid nodes can also be simple (Figure 1.32, a), or multiple (Figure 1.32, b). The multiplicity of rigid nodes is determined by the formula:

$$R = D - 1,$$

where R is the number (multiplicity) of simple rigid nodes, D is the number of disks that are monolithically connected in one node. A simple rigid connection eliminates three degrees of freedom. It is equivalent to three simple links.

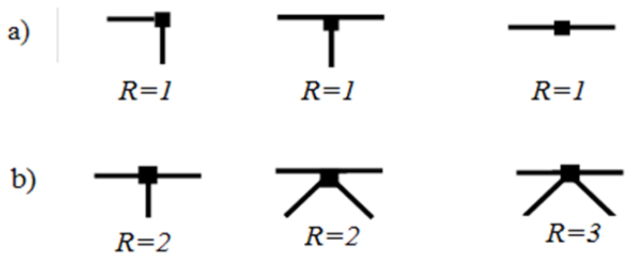


Figure 1.32

A rigid (build-in) support that eliminates the ability of the disk (bar) to move relative to the supporting surface, like a rigid node, is also equivalent to three simple links (Figure 1.7, a, b).

If necessary, rigid nodes allow breaking one disk (bar) into an arbitrary number of component bars (disks) (Figure 1.33).

Figure 1.33

If a system of disks connected by links can change the geometric shape given to it or move relative to the supporting surface, then it is a mechanism, that is, it is geometrically variable, and cannot (with rare exceptions) act as a structure.

The goal of kinematic analysis is precisely to find out:

•whether structural systems are capable of perceiving the load transferred to them without a significant change in their geometric shape,

•what should be the ratio between the number of disks and the number of constraints (links) imposed,

•what is the complexity of the calculation to determine the reactions, internal forces and displacements in the structure.

1.4.2. Degree of Freedom (Degree of Variability) of Plane Systems. Formulas for Calculating

Based on the concepts introduced above, it is easy to determine the degree of freedom $\boldsymbol{W}$ of any planar system composed of $\boldsymbol{D}$ disks connected to each other and the supporting surface by $\boldsymbol{R}$ simple rigid nodes, $\boldsymbol{H}$ simple hinges, and $\boldsymbol{L_o}$ simple support links.

If the system consists only of free, unconnected disks, then its degree of freedom will be equal to $\boldsymbol{3D}$. Each simple rigid joint introduced eliminates three degrees of freedom, each simple hinge - two, and each simple support link - one degree of freedom. Therefore, the total degree of freedom of the system is equal to the difference:

$$W = 3D - 3R - 2H - L_0. \tag{1.1}$$

For the correct application of the obtained formula, it should be remembered that $\boldsymbol{R}$, $\boldsymbol{H}$ and $\boldsymbol{L_o}$ mean the total number of, respectively, simple (single) rigid nodes, simple (single) hinged nodes and simple support links. In this case, it is necessary to ensure that each disk and each connection (each device) are counted only once. In other words, if, for example, the hinge connection of one of the disks to the ground is taken into account as a simple hinge, then this support device can no longer be included in the number of simple support links as a hinged immovable support equivalent to two support links.

The degree of freedom of a plane system, separated from supports (not having support connections), i.e., in the mounting or transport state, consists of the degree of freedom of it as a rigid whole, equal to three (on the plane) and the degree of variability of V of its elements relative to each other (internal mutability). Thus, we can write

$$W = 3 + V,$$

where from

$$V = W - 3.$$

Substituting the expression W in the last formula, provided that there are no support rods in the system, we obtain the final formula for calculating the degree of variability of the bars system disconnected from the supports

$$V = 3D - 3R - 2H - 3. \tag{1.2}$$

If the degree of freedom (or degree of variability) of the system is positive (greater than zero)

$$W > 0 \quad (\text{or } V > 0),$$

then the system is geometrically changeable. In its structure, to ensure geometric immutability, W (or V) links are missing.

For example, a suspension system (Figure 1.34) is composed of four rods connected by three hinges and is supported by two hinged immovable supports (in total 4 support rods). Its degree of freedom is equal to

$$W = 3D - 2H - L_0 = 3 \cdot 4 - 2 \cdot 3 - 4 = 2.$$

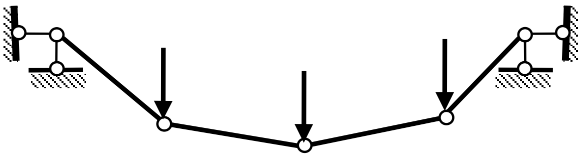


Figure 1.34

Therefore, it is geometrically changeable. Its structure lacks two links to ensure geometric immutability.

If the degree of freedom (or degree of variability) of the system is negative (less than zero) $\boldsymbol{W} < 0$ (or $\boldsymbol{V} < 0$), then the system contains an excessive number of links from the point of view of geometric immutability.

A two-span two-tier frame (Figure 1.35, a) consists of eight disk (bars). The bars are connected by two simple hinges, three double rigid nodes (six single, simple) and are supported by three absolutely rigid supports. Its degree of freedom is equal to

$$W = 3D - 3R - 2H - L_0 = 3 \cdot 8 - 3 \cdot 6 - 2 \cdot 2 - 9 = -7.$$

In terms of geometric immutability, this frame contains seven extra links.

The same frame can be considered as composed of only two disks connected by two hinges (Figure 1.35, b). One of the disks has three rigid supports (9 simple support rods). Consequently, we get the same result:

$$W = 3D - 3R - 2H - L_0 = 3 \cdot 2 - 3 \cdot 0 - 2 \cdot 2 - 9 = -7.$$

The negative degree of freedom of the system equal to the number of redundant connections determines the degree of static indeterminacy of the system. Therefore, the degree of static indeterminacy of the system can be calculated by the formula:

$$\Lambda = -W = 3R + 2H + L_0 - 3D, \qquad (1.3)$$

where Λ is the number of extra links (redundant links).

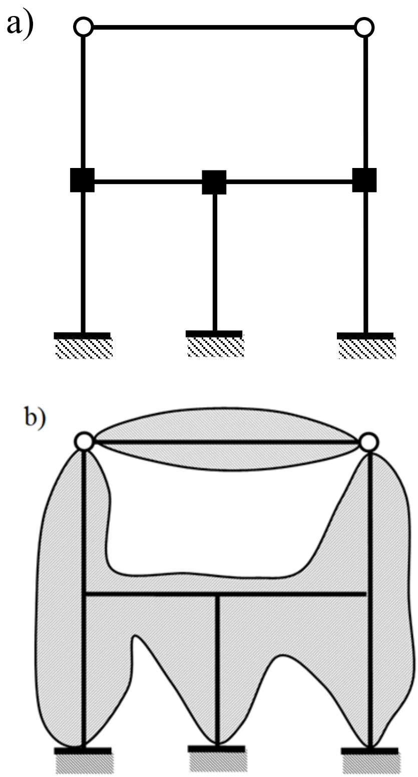


Figure 1.35

If the degree of freedom of the system is zero

$$W = 0,$$

then the system has the number of connections necessary for geometric immutability and immobility and can be statically determinate.

Such a system is shown in figure 1.36. It consists of 9 disks (bars). It has no rigid nodes. The disks are connected by 12 simple hinges (the multiplicity of hinged nodes is shown in the figure). Three supporting rods link it to the supporting surface. Its degree of freedom is equal to

$$W = 3D - 3R - 2H - L_0 = 3 \cdot 9 - 0 - 2 \cdot 12 - 3 = 0.$$

The same result can be obtained in a different way, assuming that the system is composed of 11 bars. It is assumed that both half-beams are formed by each of two bars soldered rigidly in quarters of a span. Consequently, two additional rigid nodes appear. The number of hinges and supporting rods has not changed. There are other options for calculating the degree of freedom of a given system.

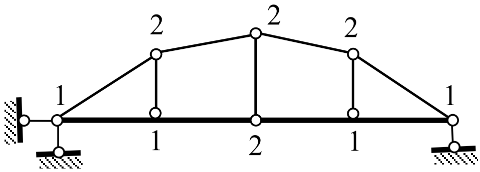


Figure 1.36

If the degree of variability of the system is zero

$$V = 0,$$

then the system has the number of bonds necessary for internal geometric immutability and can be internally statically determinate. For example, the degree of variability of a single-slope truss without supports (Figure 1.37) is zero:

$$V = 3D - 3R - 2H - 3 = 3 \cdot 13 - 0 - 2 \cdot 18 - 3 = 0.$$

The system contains the necessary number of links that are internally geometrically unchanged and statically determinate. But externally, relative to the earth, the system is mobile; it lacks at least three support connections to give it immobility. A greater number of superimposed support connections will turn it into an externally statically indeterminate system.

The calculation of the degree of freedom or the degree of variability for plane truss can also be performed using a more convenient formula.

In the truss, the hinged nodes can be considered as material points having two degrees of freedom on the plane. The truss rods, as well as the support rods, can be considered as simple links.

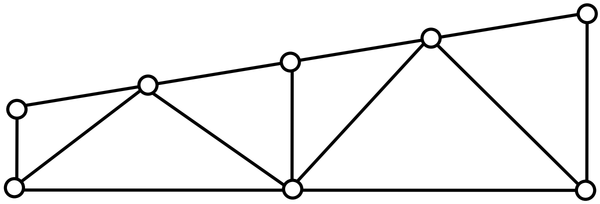


Figure 1.37

If the nodes of the truss were not connected by rods, then the system of ***N*** free nodes would have *2N* degrees of freedom. The truss rods connecting the nodes and the support rods, each as a simple link, eliminate one degree of freedom. Therefore, the degree of freedom of the plane truss can be calculated by the formula

$$W = 2N - B - L, \tag{1.4}$$

where N – the number of truss nodes as material points,
B – the number of rods of the truss,
L – the number of support rods (simple links).
Accordingly, the degree of variability of the truss disconnected from the supports will be equal to

$$V = 2N - B - 3. \tag{1.5}$$

So for a farm without supports (Figure 1.37) we have

$$V = 2 \cdot 8 - 13 - 3 = 0.$$

Thus, the use of the above formulas to calculate the degree of freedom or the degree of variability of plane bars systems provides the necessary analytical criteria for geometric immutability or variability, static definability or indeterminacy.

Unfortunately, these analytical criteria are necessary, but not sufficient.

1.5. Geometrically Unchangeable Systems. Principles of the Formation

The above formulas for calculating the degree of freedom (degree of variability) of bars systems provide only a formal assessment of the kinematic properties of the systems under study, which is not always true. For the final conclusion about the geometric immutability and static definability of the bar system, an analysis of its structure, an analysis of the principles by which it is assembled is necessary. Only systems of the correct structure will be truly geometrically unchangeable.

For example, a system being partially statically indeterminate and partially geometrically variable (Figure 1.38) refers to systems of irregular structure, although its total degree of freedom is zero. The system shown in Figure 1.39 also has a zero degree of freedom, but in fact it is instantaneously changeable, since it has infinitely small mobility. Its structure is also irregular. An instantaneously rigid system (Figure 1.40) formally has one degree of freedom, but in fact it has two degrees of freedom. In addition, it can have initial efforts (for example, from cooling its elements), as once a statically indeterminate system.

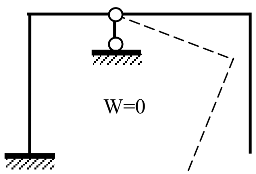


Figure 1.38

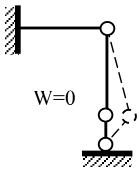


Figure 1.39

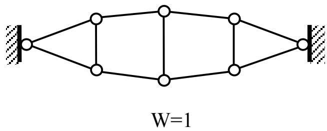


Figure 1.40

For systems of irregular structure, the concepts of the degree of freedom or the degree of variability, calculated by the formulas derived above, become indefinite, meaningless.

Let us consider the main methods for the formation of obviously geometrically unchangeable bar systems.

1. The dyad method. The degree of freedom of the system (disk) will not be changed if you attach (disconnect) the hinge node using two hinged rods not lying on one straight line (Figure 1.41). Disks and any other subsystems that are known to be statically definable and geometrically unchangeable (Figure 1.42) can act as such rods.

2. The triangles method. Three disks 1, 2 and 3 connected by three hinges A, B and C, not lying on one straight line (Figure 1.43), form a new internally geometrically unchangeable system (disk). The total number of extra links, if they are in the source disks, is not changed. The total degree of freedom of the three discs is reduced by six units.

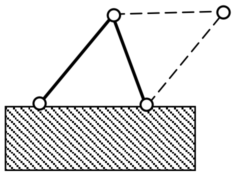


Figure 1.41

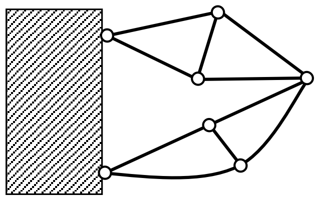


Figure 1.42

3. The method of hinge and simple link, equivalent to the method of triangles. Two disks 1 and 2, connected by a common hinge C and one rod AB, provided that the straight line AB (or its extension) does not pass through the hinge C, form a new single disk (Figure 1.44). At the same time, the total number of extra links in the source disks does not change, and their total degree of freedom is reduced by three units.

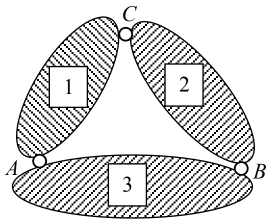


Figure 1.43

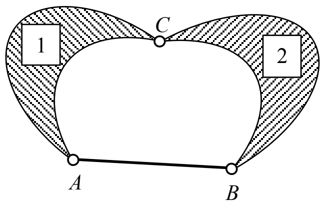


Figure 1.44

4. The three links method. Two disks are connected by three hinged rods (Figure 1.45), lying on straight lines that are not intersected at one point and are not parallel to all three at once, form a united system (new disk). In the new system, the total number of excess links, if they were in the original disks, does not change, and the total degree of freedom is reduced by three units.

Generally speaking, the considered methods of forming a single system of several components are applicable to any system with redundant links (statically indeterminate disks), and to systems with missing links (mechanisms).

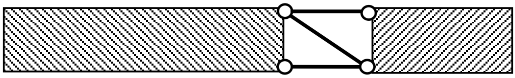


Figure 1.45

In order for a united system to be formed according to the considered laws to be geometrically unchangeable and statically determinate, it is necessary and sufficient for its components, each separately, to be geometrically unchangeable and statically determinate. Moreover, each disk can be considered as a rod and each rod can be considered as a disk. Then the considered methods of formation of obviously geometrically unchangeable and statically determinate systems can be reduced to two main methods.

1. The triangles method: three disks (rods) connected by three hinges that do not lie on one straight line form a deliberately geometrically unchangeable (internally) and statically determinate system (new disk) (Figures 1.43, 1.44).

2. The three connections method: two disks connected by three hinged rods whose axes do not intersect at one point (three parallel rods can be considered intersecting at infinity), form a new disk (Figure 1.45).

Certainly, the considered methods of formation, assembling (or dismantling, disassembling) of obviously geometrically unchangeable and statically determinate systems can be applied not only individually, but also in their arbitrary combination, sometimes replacing each other.

So, a three-hinged arch with a tie-bar (Figure 1.46) can be considered as formed:

• By the dyad method. Firstly the support hinge A is unmovably attached to the ground using the two hinged support rods. Secondly the support hinge B is fixed by the third support rod and the bar AB. Finally, the hinge C is made immovable by means of two half-arches.

• By the triangle method. The support rods of the support A, together with the ground, form the first triangle and the first single disk. The resulting disk, the beam AB and the support rod of the support B form a new single disk. Finally, the disc AB and the semi-arches AC and BC form the resulting triangle disc ABC.

• The combination of the three connections method and the method of dyads (or triangles). Beam AB is connected to the ground by three simple links (support rods). The hinged node C is attached to the resulting system by the dyad method (or a triangle ABC is formed).

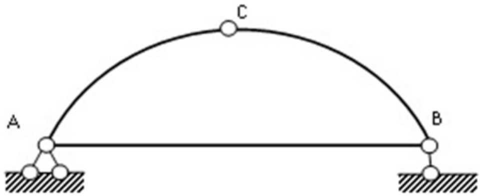


Figure 1.46

Kinematic analysis of already created system can be carried out in the reverse order, i.e., by dismantling. If, as a result of discarding nodes and bars (disks) connected according to the rules considered above, there remains a known geometrically unchangeable and statically determinable subsystem, or only one supporting surface, then the original system is geometrically unchangeable and statically determinable.

Using the analysis of the structure (analysis of the order of formation) of the system, it is easy to establish in which part of the system there are redundant links and in which part of the system they are lacking. Thus, systems of irregular structure and systems with degenerate configurations can be revealed.

Any system in a degenerate configuration, instantly changeable or instantly rigid, can be considered both statically indefinable and geometrically changeable. The structure of such systems lacks connections in one direction and at the same time there are redundant connections in other directions.

It is the presence of superfluous links that gives the degenerate system the properties of a statically indeterminate system, namely: the ability to have initial internal forces in the absence of load. And this property leads to a static criterion for instantaneous variability or instantaneous rigidity.

1. If in a system with a zero degree of freedom ($W = 0$), i.e. in a system formally geometrically unchangeable and statically determinate, there may be initial internal forces (forces due to prestressing), then such a system is instantaneously changeable or partially statically indeterminate, and partially geometrically changeable. In the latter case, it is necessary to conduct a kinematic analysis of the system by fragments.

2. If in a system with a positive degree of freedom ($W > 0$), i.e. in a system formally geometrically changeable, there may be initial internal forces (prestressing forces), then such a system is instantaneously rigid or has statically indeterminate fragments in its composition.

The connections in such systems, from the point of view of geometric changeability and mobility, are not arranged correctly.

For example, in an instantaneously changeable system (Figure 1.47), the node C is fastened from horizontal displacement by the bar AC. The bar BC also eliminates the horizontal displacement of the node C and is redundant. At the same time, there is no any link in the system that would eliminate the vertical displacement of the node C. However, such an offset can only be infinitesimal: as soon as the node C moves off the line AB, the dyad bars AC and BC will no longer lie on one straight line and further displacement of the C node will become impossible without deformation of the AC and BC bars. From a static point of view, in this system initial forces without load are possible, for example, due to cooling or displacements of supports.

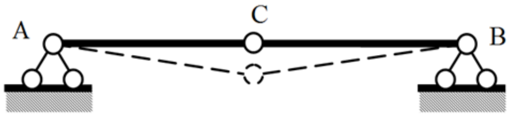


Figure 1.47

In the cable truss (Figure 1.40) in the middle panels, from the point of view of its formation by the method of triangles, two diagonal bars are clearly absent. Therefore, this truss must have two degrees of freedom. At the same time, it has four support bars, one of which (horizontal) is superfluous. Total degree of freedom $W = 1$. But precisely because of the presence of this extra connection (one of the horizontal support bars) in a given geometrically changeable system, only infinitely small displacements are possible. From a static point of view, this system at $W=1>0$ also allows preliminary tension. This means that this system is instantaneously rigid.

A disk connected to the support surface by three support rods formally should have a zero degree of freedom. But if the three support rods converge in one support hinge (Figure 1.48), the system will remain geometrically changeable (there is freedom of rotation about the axis of the support hinge), while the hinged immovable support has an extra (for a plane case) support rod.

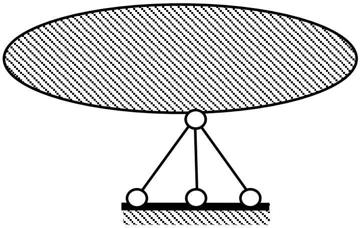


Figure 1.48

The hinge-rod disc DFB (Figure 1.49), formed by the method of triangles, is connected to the fixed points A and C by the L-shaped rods AD and CF and is supported by hinged movable support B with the vertical support rod, i.e. it is connected to the supporting surface by three rods-discs ($W = 0$). But the lines on which the ends of these three rods-disks lie intersect at one point O, which is the center of instant rotation. Initial efforts are possible in the system due to jacking up of the central support. Therefore, this system is instantaneously changeable.

Examples of some other systems of irregular structure are shown in Figure 1.50 (the system is geometrically variable, though $W = -2$) and in Figure 1.51 (a system with a statically indeterminate fragment is instantaneously changeable at $W = -3$).

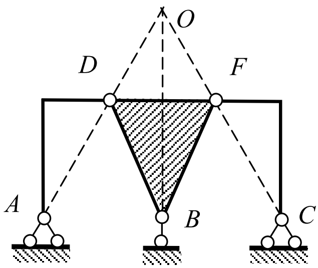


Figure 1.49

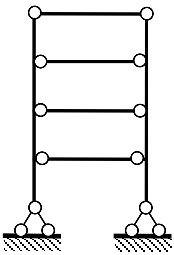


Figure 1.50

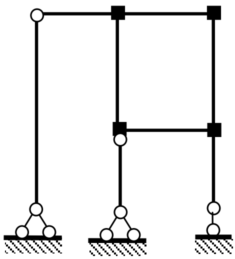


Figure 1.51

1.6. Matrices in Problems of Structural Mechanics

When carrying out calculations based on computer technology, discrete schemes of structures and matrix calculus methods are used in structural mechanics. The loads acting on the structure are represented in the form of a load vector (matrix-column), the components of which are the values of the specified loads, numbered in a certain order. The calculation results will be presented not in the form of diagrams of internal forces or displacements, but in the form of force vectors and displacement vectors, in which the values of internal forces in specific sections and the values of displacements of specific points in given specific directions will be listed.

So the loads applied to a simple beam (Figure 1.52) can be represented by a third-order vector

$$\vec{F}_1 = [q_1 \;\; F_2 \;\; M_3]^T,$$

and the loads applied to the beam truss (Figure 1.53), by a fifth-order vector

$$\vec{F}_1 = [F_1 \;\; F_2 \; ... \; F_5]^T.$$

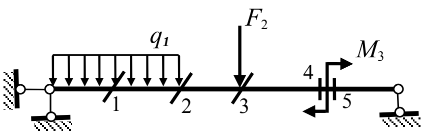


Figure 1.52

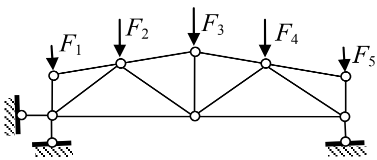


Figure 1.53

To find bending moments in five characteristic sections of the beam (Figure 1.52) and internal forces in thirteen rods of the truss (Figure 1.53) from the given loads, it is enough to construct, respectively, the influence matrix of bending moments L_M for the beam and the influence matrix of longitudinal forces L_N for the truss, the rods of which must be numbered beforehand. Then use the matrix formulas

$$\vec{M} = L_M F_1, \quad \vec{N} = L_N F_2,$$

where

$$\vec{M} = \begin{bmatrix} M_1 \\ M_2 \\ \dots \\ M_5 \end{bmatrix}, \qquad L_M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ \dots & \dots & \dots \\ m_{51} & m_{52} & m_{53} \end{bmatrix},$$

$$\vec{N} = \begin{bmatrix} N_1 \\ N_2 \\ \dots \\ N_{13} \end{bmatrix}, \qquad L_N = \begin{bmatrix} n_{11} & n_{12} & \dots & n_{15} \\ n_{21} & n_{22} & \dots & n_{25} \\ \dots & \dots & \dots & \dots \\ n_{13,1} & n_{13,2} & \dots & n_{13,5} \end{bmatrix}.$$

The element m_{ik} of the influence matrix of bending moments is a bending moment in a characteristic beam section number i, caused by a unit load number *k*. The element n_{ik} of the influence matrix of the longitudinal forces is the force in the rod number ***i*** of the truss from a unit value of the external force $F_k = 1$.

Using a suitably constructed an influence matrix of displacements *D*, we can find the vector $\vec{\Delta}$ of displacements of given points in given directions due to external forces given by the vector $\vec{F}$:

$$\vec{\Delta} = D\vec{F},$$

where

$$\vec{\Delta} = \begin{bmatrix} \Delta_{1F} \\ \dots \\ \Delta_{nF} \end{bmatrix}, \qquad D = \begin{bmatrix} \delta_{11} & \dots & \delta_{1k} \\ \dots & \dots & \dots \\ \delta_{n1} & \dots & \delta_{nk} \end{bmatrix}, \qquad \vec{F} = \begin{bmatrix} F_1 \\ \dots \\ F_n \end{bmatrix}.$$

The symbol Δ_{nF} denotes the displacement of a point (section) number *n* in the direction of the force $F_n = 1$ applied at this point, caused by a given load. The element δ_{nk} of the influence matrix of displacements *D* is equal to the displacement of a point (section) number *n* in the direction of the force $F_n = 1$ caused by the force $F_k = 1$, and is called the unit displacement.

Thus, the use of influence matrices is based on the principle of independence of the action of forces, the principle of superposition. According to this principle, the total effect of several forces is equal to the sum of the effects of each force individually. At the first stage, the calculation is reduced to the calculation of internal forces and displacements from a single external forces and the construction of influence matrices. At the second stage, using the matrix formulas forces and displacements from any combination of loads are calculated with the help of computer.

The displacement influence matrix ***D*** is also called the flexibility (compliance) matrix. The flexibility matrix allows you to express displacements through external forces. The square flexibility matrix can be inverted and a new matrix *R*, which is called the stiffness matrix, can be obtained:

$$R = D^{-1}.$$

The stiffness matrix allows you to express external forces through the displacements of points to which these forces are applied

$$\vec{F} = R\vec{\Delta}.$$

Without going into detail we note that the flexibility and stiffness matrices are widely used in the analyses of statically indeterminate systems, as well as in the dynamics and stability of structures. On the basis of matrix calculus, modern design and computing complexes have been created for analyzing structures using computers.

THEME 2. STATICALLY DETERMINATE SYSTEMS. MAIN CHARACTERISTICS. ANALYSIS METHODS UNDER FIXED LOADS

2.1. Concept of Statically Determinate Systems. Main Characteristics

One of the main tasks of structural mechanics is to determinate the internal forces in the elements of a structure. The methods for their determination depend on those assumptions that are accepted for calculation. The division of systems into statically determinate and statically indeterminate depends on these assumptions. According to some assumptions, the same design scheme is considered to be statically determinate, while the others describe it as statically indeterminate.

With a strict formulation of the calculation problem, it is necessary to define the internal forces taking into account the deformable state of the structure. In this case, as a rule, all systems are statically indeterminate.

In a real linearly deformable system, deformations and displacements are small. Their influence on the distribution of internal forces is neglected. The calculation is carried out according to the so-called undeformed design scheme. It is assumed that the geometry of the deformed structure coincides with the geometry of the original undeformed structure.

Statically determinate systems are those systems in which all internal forces can be determined **only from equilibrium equations.**

The main properties of statically determinate systems are the following:

1. A statically determinate system has no redundant constraints (links), i.e. $W = 0$. When at least one link is removed; the statically determinate system becomes a geometrically changeable system.

2. Internal forces in statically determinate systems are independent of the elastic properties of the material and the dimensions of the cross sections of the elements.

3. Changes of temperature, settlements of supports, slight deviations in the lengths of the elements do not lead to occurrence of additional forces in a statically determinate system.

4. A given load in a statically determinate system corresponds to one single possible picture of the distribution of internal forces.

5. The self-balanced load applied to the local part of the system causes an appearance of internal forces in the elements of that part only. In the remaining elements of the system, the internal forces will be zero (Figure 2.1).

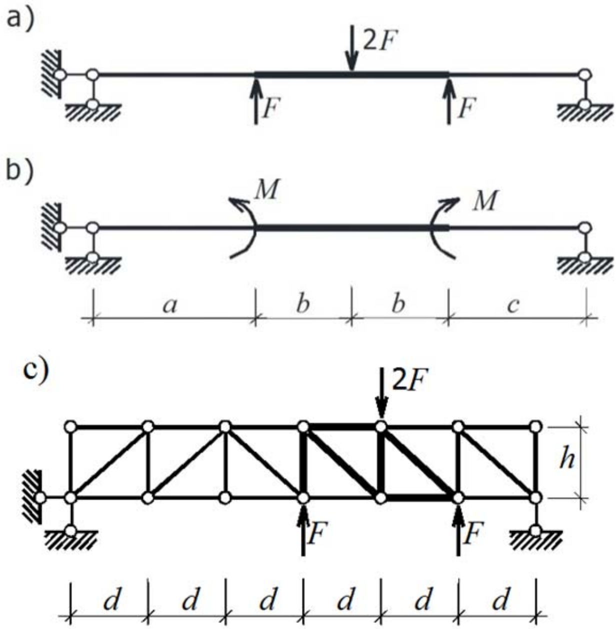


Figure 2.1

2.2. Sections Method

A bending moment (M), longitudinal (N) and transverse (Q) forces, which are internal forces in a cross section of an element of a plane system, can be integrally expressed through normal (σ) and tangential (τ) stresses (Figure 2.2).

The sign of the bending moment M depends on the sign of curvature of the bended bar and the selected direction of the axes of the external fixed coordinate system (Figure 2.3). If the axis is directed in the opposite direction, then the curvature sign, and hence the moment sign, will be reversed.

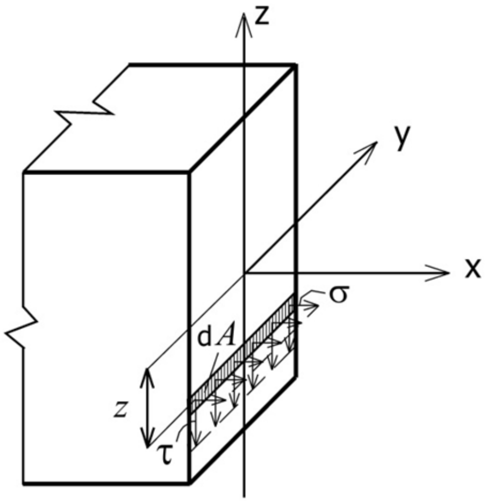


Figure 2.2

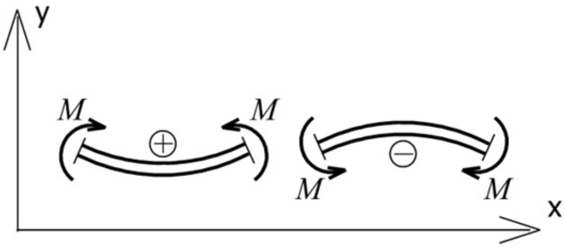


Figure 2.3

When constructing bending moment diagrams, the positive ordinate of the moment is drawn in the direction of convexity of the bended axis, i.e. the diagram of moments is plotted on the stretched fibers of the element.

The transverse force is considered positive if it tends to rotate the cut off part of the bar clockwise (Figure 2.4, a). The bar parts separated by the cross section are spaced apart in Figure 2.4.

Longitudinal force is considered positive if it causes stretching of the bar (Figure 2.4, b).

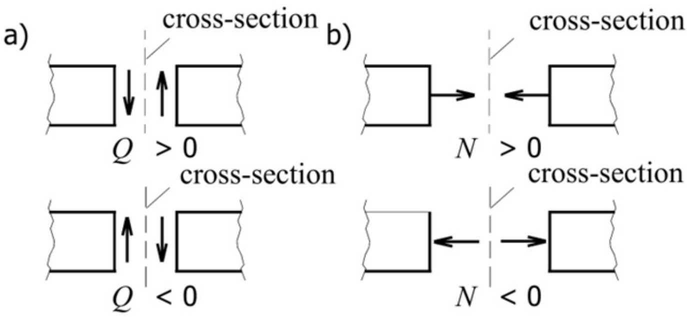


Figure 2.4

To determine the internal forces M, Q and N, equilibrium equations are used, which can be written in one of three forms:

1. The sum of the projections of all the forces on each of the two coordinate axes and the sum of their moments relative to any point C_1 lying in the plane of the forces must be equal to zero:

$$\sum X = 0, \ \sum Y = 0, \ \sum M_{C_1} = 0.$$

2. The sums of the moments of all forces relative to any two centers C_1, C_2 and the sum of the forces projections onto any axis X not perpendicular to the line $C_1 C_2$ should be equal to zero:

$$\sum X = 0, \ \sum M_{C_1} = 0, \ \sum M_{C_2} = 0.$$

3. The sums of the moments of all forces relative to any three centers C_1, C_2 and C_3, not lying on one straight line, should be equal to zero:

$$\sum M_{C_1} = 0, \ \sum M_{C_2} = 0, \ \sum M_{C_3} = 0.$$

The ways of using these equations to determine the internal forces depend on a given system structure.

When using the way of simple sections, at first, the studied system is divided into two independent parts by the section in which the internal forces must be determined, and then the action of one part by the other is replaced by the desired internal forces. To determine them, the equilibrium equations are compiled (in any of the forms listed above). Then these equations are solved, provided, that the support reactions of the studied system are calculated in advance. For example, determining the efforts in the frame cross-section ***k*** (Figure 2.5, a), we can consider the equilibrium of the right-hand part of the frame (Figure 2.5, b) and make equations:

$$\sum X^{(right)} = F_3 - N_k = 0;$$

$$\sum Y^{(right)} = V_B - F_2 + Q_k = 0;$$

$$\sum M_k = V_B\, b - F_2\, b_1 + F_3\, h_2 - M_k = 0.$$

Having solved them, we define the efforts N_k, Q_k and M_k. A positive sign of the found force indicates that the given direction of the force is valid.

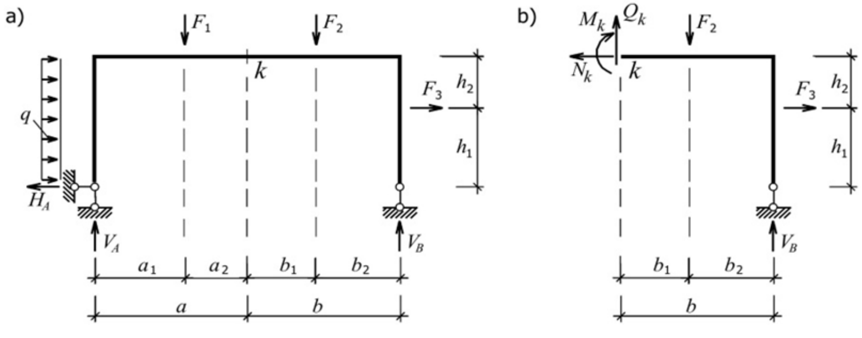


Figure 2.5

When choosing the form of the equilibrium equations should strive to ensure that the problem is solved in a most simply way: each equation, if possible, should contain only one unknown force.

Using the methods of forming geometrically unchangeable systems (see Theme 1), the rigid connection of the left and the right parts of the frame, for example, in the cross-section ***k*** (Figure 2.5, a) can be represented in a discrete view, i.e. in the form of some simple links. With a certain positions of links in the cross-section, the force in a single link (link reaction) will be equal to the corresponding internal force, i.e. N_k, Q_k or M_k.

Possible variants of the links location in the cross-section k are shown in figures 2.6, a, …, c. The efforts in the links that correspond to the required internal forces are also indicated there.

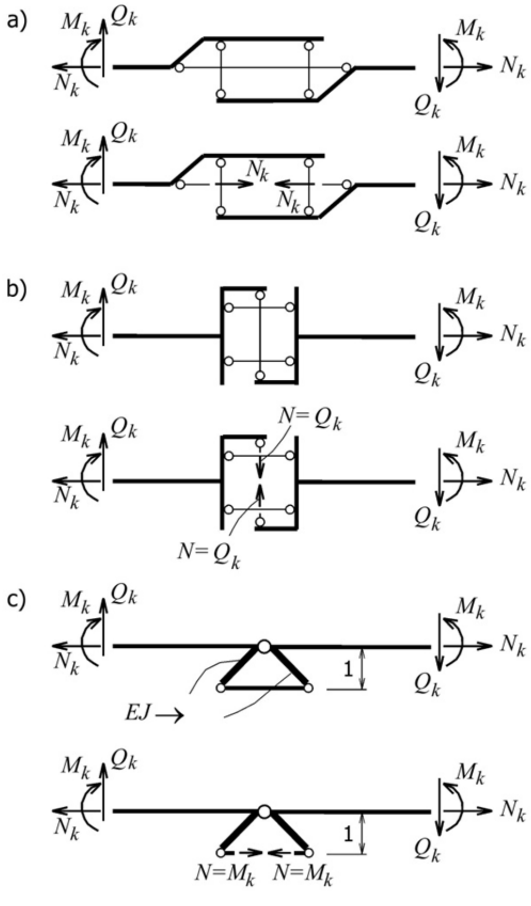


Figure 2.6

In this way, any rigid cross-section of a solid rod can be considered as a rigid node connecting two parts of a structure. Such a rigid node can always be approximated by three simple links. This approximation is used for determining internal forces by static and kinematic methods, for constructing influence lines for internal forces, and for other problems.

A variation of the static method for determining efforts is the way of dividing the system under study into many separate fragments. Composing equilibrium equations for each of them, taking into account, of course, internal forces (they are unknown) in the cross-sections separating fragments, we obtain for a statically determinate system a complete system of equations, the solution of which gives values of unknowns.

We divide, for example, the frame (Figure 2.7, a) into three fragments, shown in Figure 2.7, b. The total number of unknowns is nine: four support reactions, three unknowns in cross section D and two in cross section C. For each of the three fragments (disks), we can create three independent equations in any of the previously listed forms. Solving a joint system of linear equations of the 9th order will enable us to find all the unknowns.

Further expansion of this method of calculating efforts is associated with the division of a given system into separate elements and nodes. Read about it in the textbook (theme 15).

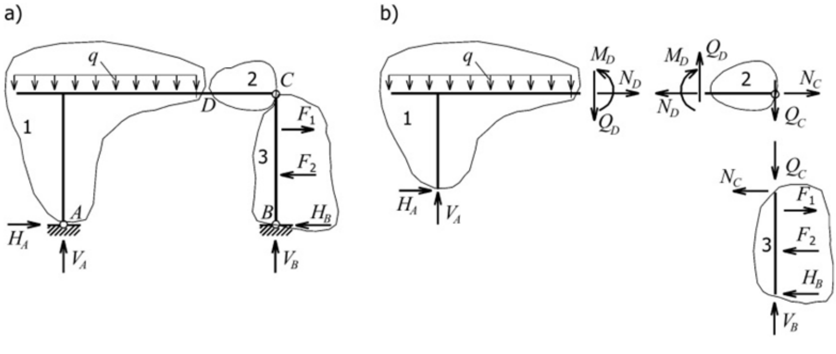


Figure 2.7

2.3. Links Replacement Method

Consider the application of this method to the calculation of the truss, shown in Figure 2.8, a.

The truss is statically determinate. Its structure can be represented in the form of three disks (triangles 3-5–6, 4–6–7 and rod 1–2), pairwise connected by two links. Since the intersection points of rods 1–3 and 2–5, 2–4, and 1–7 and node 6 (poles of mutual rotation of the disks) do not lie on one straight line, the truss is not instantaneously changeable structure.

A truss cannot be calculated by nodes isolation method, without solving the system of equilibrium equations for all nodes. It is also impossible to apply the method of simple sections, since there is no section dividing the system into two parts, in which there will be no more than three unknown forces.

The essence of the links replacing method is that one of the links of a given system is removed, and its action is replaced by an unknown force. In order for the system to remain geometrically unchangeable, another link is introduced into it. With a good arrangement of this connection, the new system (it is called a replacing system) is simpler to analyze. Static equivalence of the given and replacing systems will be observed when X becomes equal to the true force in the selected rod. In this case, the reaction in the introduced additional link will be equal to zero. Zero effort in an additional connection is a condition for writing an equation from which the force X is determined.

Let consider at an example. In a given truss (Figure 2.8, a), we will remove rod 1–2, and its effect on nodes 1 and 2 will be replaced by forces X_1. We introduce an additional link (support) in the sixth node. The replacing system obtained by such transformations is shown in Figure 2.8, b. The efforts in its rods are easily determined by the nodes isolation method.

Performing its calculation, we use the forces superposition principle. First we find the forces in the rods when loading the system with a given external load (Figure 2.8, c). We will denote them $N_{i-k,F}$. The force in the additional support connection – R_{1F} (index 1 means the number of the additional connection, the index F indicates the cause of the force). For the sizes adopted in Figure 2.11, a, we obtain $R_{1F} = 0.4023 \cdot F$.

Let us calculate the replacing system for the action $X_1 = 1$ (Figure 2.8, d). The efforts in the rods will be denoted $N_{i-k,1}$. The force in the additional connection – r_{11} (the first index, as before, is the number of the additional connection; the second indicates the reason that caused the effort). In the case under consideration this reaction is equal to $r_{11} = 0.1380$.

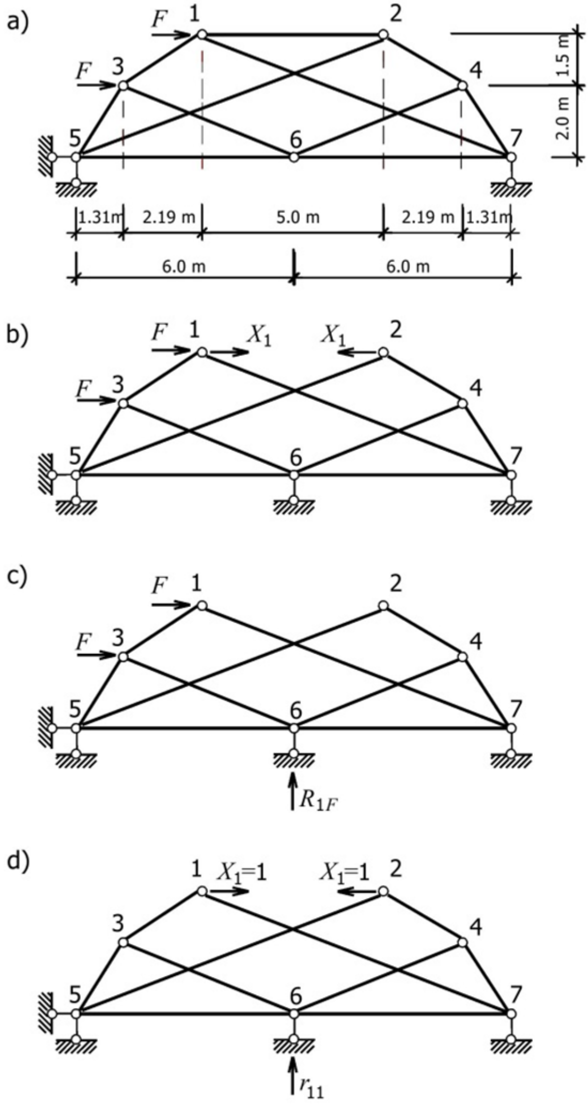


Figure 2.8

Since the total reaction of the additional support is equal to zero, we can write the equation

$$r_{11}X_1 + R_{1F} = 0, \tag{2.1}$$

from which we find

$$X_1 = -\frac{R_{1F}}{r_{11}} = -2.915F.$$

If it turned out that $r_{11} = 0$, then this would be a sign that the given truss is instantaneously changeable structure.

Subsequent calculation of the truss can be performed by nodes isolation method, or, if all $N_{i-k,F}$, $N_{i-k,1}$ are known, the forces in the rods of a given truss can be calculated by the formula

$$N_{i-k} = N_{i-k,F} + N_{i-k,1}X_1.$$

Let us consider another example. A multi-span beam (Figure 2.9, a) is easily calculated by the simple section method. However, in order to better understand the essence of the links replacement method, we will show its calculation with this method.

In the given beam, we remove the support connections at the points B and D. Their action on the beam is replaced by forces X_1 and X_2. Let us introduce additional moment links at the points A and C, i.e. close the hinges. The replacement system obtained by these transformations is shown in Figure 2.9, b or, in a more familiar image form, in Figure 2.9, c.

Let us construct the bending moment diagrams in the replacing beam caused by given load (Figure 2.9, d), unit force X_1 (Figure 2.9, e) and unit force X_2 (Figure 2.9, f). The values of the moments in additional constraints caused by these loads are shown in the figures.

From the conditions of static equivalence of the given and replacing beams it follows that the forces (moments) in the first and second additional links must be equal to zero. Defining them according to the principle of independence of the action of forces, we obtain the following system of equations:

$$\left.\begin{aligned} r_{11}X_1 + r_{12}X_2 + R_{1F} &= 0; \\ r_{21}X_1 + r_{22}X_2 + R_{2F} &= 0. \end{aligned}\right\} \tag{2.2}$$

Let us write the equations in numerical form:

$$\left.\begin{aligned} 4X_1 + 9X_2 - 80 = 0; \\ 4X_2 - 10 = 0. \end{aligned}\right\}$$

Solving them, we find $X_1 = 14.375\, kN$, $X_2 = 2.5\, kN$.

The diagram of moments for a given beam is constructed by the expression

$$M = M_F + M_1X_1 + M_2X_2.$$

It is shown in Figure 2.9, g.

It is clear that in general, the number of deleted and additional links can be large.

Let us write the system of equations (2.2) in matrix form:

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} R_{1F} \\ R_{2F} \end{bmatrix} = 0, \quad \text{or} \quad L\vec{X} + \vec{R}_F = 0. \tag{2.3}$$

The solution of system (2.3), written in the form

$$\vec{X} = -L^{-1}\vec{R}_F, \tag{2.4}$$

possible if only the determinant of the matrix L is not equal to zero:

$$Det\ L \neq 0.$$

Therefore, if the determinant is equal to zero:

$$Det\ L = 0,$$

it serves as a sign of instantaneously changeability of a given system.

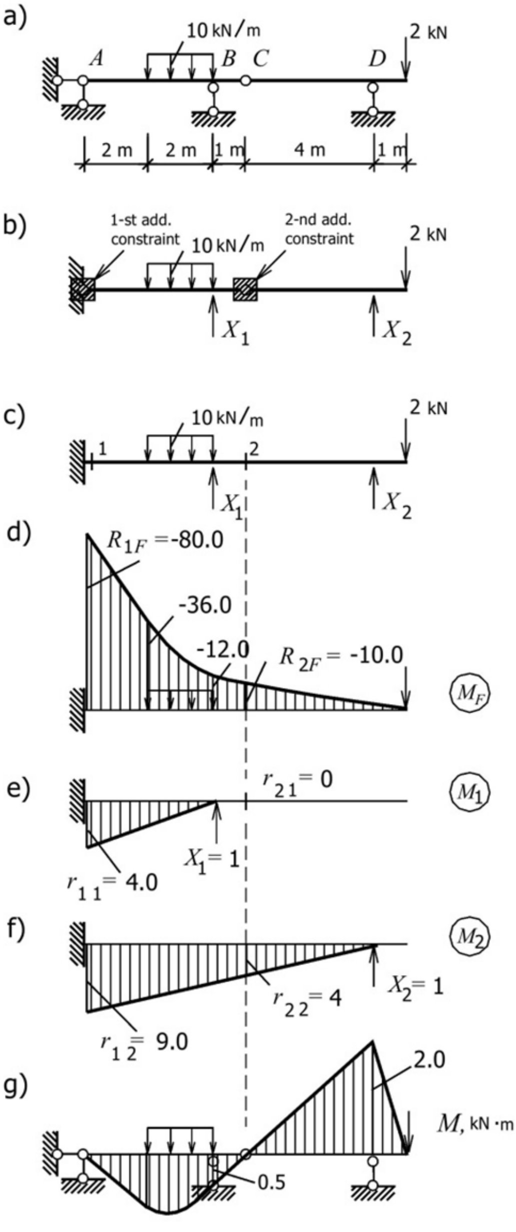

a)
10 kN/m
2 kN
A
B
C
D
2 m
2 m
1 m
4 m
1 m
b)
1-st add. constraint
10 kN/m
2-nd add. constraint
2 kN
X_1
X_2
c)
10 kN/m
2 kN
1
2
X_1
X_2
d)
R_{1F} =-80.0
-36.0
-12.0
R_{2F} = -10.0
M_F
e)
r_{21}= 0
M_1
X_1= 1
r_{11}= 4.0
f)
M_2
X_2= 1
r_{22}= 4
r_{12}= 9.0
g)
2.0
M, kN·m
0.5

Figure 2.9

2.4. Kinematic method

The kinematic method is based on the principle of virtual displacements, which allows to obtain the necessary conditions for the equilibrium of the system.

Virtual displacements of a system are any combinations of infinitesimal displacements of points of a system allowed by its connections. Virtual displacements, unlike real ones, do not depend on the given external actions. They are determined only by the type of system itself and the type of connections superimposed on the system; these are purely geometric concepts.

We assume that during the transition of the system from the real state to the new one, caused by virtual displacements, the external and internal forces do not change.

The work of external and internal forces performed on virtual displacements is called the virtual work. Taking into account the introduced remarks, this work is defined as the work of constant forces on virtual displacements.

The principle of virtual displacements establishes the general condition for the equilibrium of the deformed system. It is formulated as follows if the system is in equilibrium under the action of external forces applied to it, then for any infinitely small virtual displacements of the points of this system, the sum of the works of its external and internal forces is zero. Let us show a formal record of this principle in the form:

$$W^{(virt)} + A_{\text{int}}^{(virt)} = 0, \tag{2.5}$$

Where $W^{(virt)}$ – virtual work of external forces,

$A_{\text{int}}^{(virt)}$ – virtual work of internal forces.

Introducing the concept of the degree of freedom of the rod system (Sec. 1.4), we assumed that its rods are absolutely solid, non-deformable. Given this assumption, and also taking into account the concept of virtual displacements, it should be noted that in the initial state for a statically determinate system (W = 0) it is impossible to specify virtual displacements. How is then to apply the principle of virtual displacements to the calculation of such systems?

To use this principle in the problems of calculating statically determinate systems, the main axiom of the mechanics of non-free material bodies are applied – the principle of removing constraints (links). Let us remove any constraint (support, or from among those shown in Figure 2.6) and apply to the system, in addition to the given external forces, the force S that could occur in the removed constraint. Such a system will be a mechanism with one degree of freedom (W = 1) and, therefore, allows a possible new position, determined by one parameter. Its equilibrium state is possible only if the unknown force S in the remote constraint is equal to the true value.

Let us provide the principle of virtual displacements to the mechanism received. The work of internal forces along the entire length of non-deformable elements is zero. Considering the effort in the removed constrain as an external force, the equation of virtual works of all forces can be written as:

$$W^{(virt)} = S_i \delta_i + \sum F_k \Delta_k = 0, \tag{2.6}$$

where S_i – is the required effort in connection i, δ_i – is displacement in its direction;

F_k – k - th generalized force, Δ_k – displacement in the direction of the force F_k.

If the direction of the force and the corresponding displacement coincide, then the work is positive.

Since the calculation is carried out according to an undeformed scheme, in the system with one degree of freedom, all displacements δ_i and Δ_k are expressed in terms of one parameter. Having divided each term of equation (2.5) by this parameter, we solve it relatively S_i.

For example, determining the reaction V_B in the support B of a two-span statically determinate beam (Figure 2.10, a), we remove the support rod at a point B and apply an unknown force V_B at this point. The position of the mechanism with one degree of freedom is determined by one parameter. To such parameter, we take the rotation angle φ of the beam AB (Figure 2.10, b). Since φ, by the definition, is an infinitesimal angle, then $\Delta_1 = 2l\varphi$, $\Delta_B = 4l\varphi$, $\Delta_2 = 5l\varphi$, $\Delta_3 = 5l\varphi$.

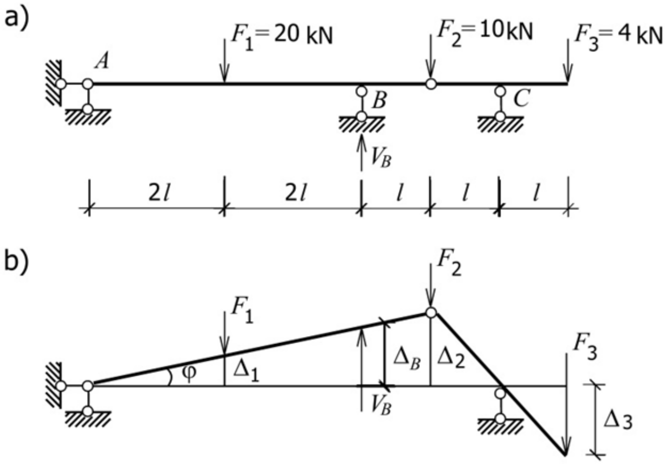


Figure 2.10

The equation of works (2.5) can be written as:

$$W^{(virt)} = V_B 4l\varphi - F_1 2l\varphi - F_5 5l\varphi + F_3 5l\varphi = 0.$$

The solution gives $V_B = 17.5\,kN$.

When determining the force in the rod 1–2 of the truss beam (Figure 2.11, a), the sequence of actions remains the same as in the previous example. By removing the rod 1–2 in the given beam we get the mechanism. Virtual displacements of the mechanism will be set as follows. Keeping point *C* stationary, move support *B* vertically. In this case, the bar *CB* rotates by an infinitesimal angle φ (Figure 2.11, b). Considering the known support reaction as an external force, we compose the equation of virtual work.

From the equation of virtual work

$$-N_{1-2} 2\varphi + V_B 4\varphi - q\,\frac{1}{2}\,2\;2\varphi = 0$$

we find $N_{1-2} = 35\ kN.$

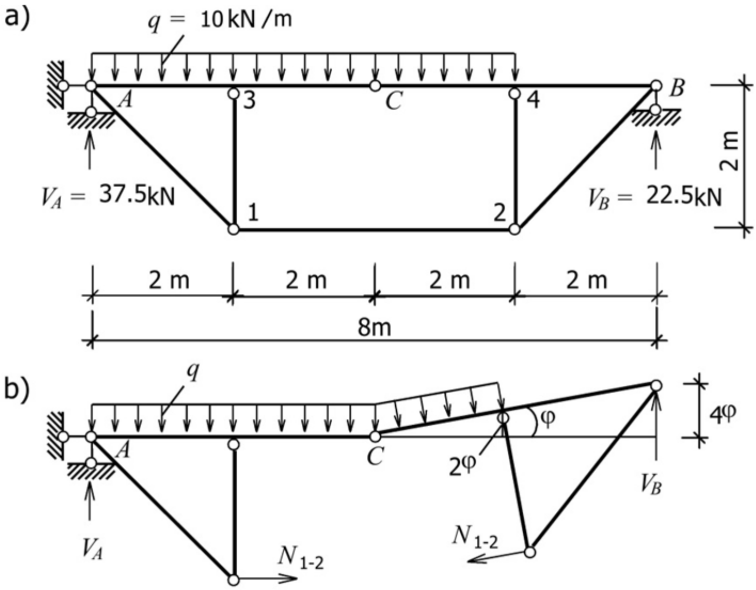


Figure 2.11

2.5. Statically Determinate Multi-Span Beams and Compound Frames. Main Characteristics

Statically determinate multi-span beams are a collection of simple beams connected to each other at the ends by hinges, as a rule, not coinciding with the supports.

Before starting the calculation of a multi-span beam, it is necessary to control its geometric changeability.

Kinematic analysis of multi-span beams is performed according to the rules outlined in Theme 1. After checking the degree of freedom according to formula (1.1), you should analyze the interaction scheme of simple beams in a multi-span structure (analyze the structure of the system). To do this, mentally divide the multi-span beam (Figure 2.12, a) through the hinges and analyze each simple beam for changeability. The beam AB is fixed by three correctly located support rods (links); this beam is unchangeable. It may be called the main beam or primary ones.

Then, the state of the beam adjacent to it on the right side is considered. This beam CD has its own vertical support link at point D. The hinge, which connects the beams at point C, can be replaced by two support links. We draw (Figure 2.12, b) the position of the beam CD above the main one (gravity is transmitted from upper beam to lower one). A beam CD will be called an auxiliary beam or secondary one. Considering in the same way, we show the position of the upper auxiliary beam EF. The design scheme shown in Figure 2.12, b is called interaction scheme.

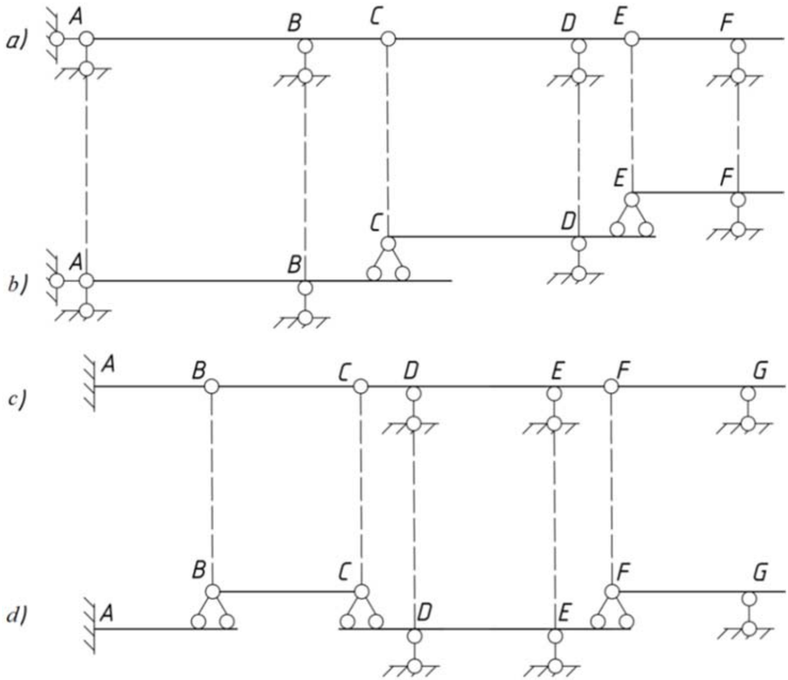


Figure 2.12

Interaction schemes for multi-span beams can be varied. As an example, figure 2.12, d shows the interaction scheme for a multi-span beam in figure 2.12, c. There are two main beams AB and DE. Beams BC and FG are auxiliary.

Using the interaction schemes, the sequence for calculating a multi-span beam is established. First, the uppermost auxiliary beams are calculated, then below located beams are analyzed taking into account the interaction forces (pressure from the upper beams is transmitted to the lower beams).

E x a m p l e. We perform a kinematic analysis and show the sequence of plotting the bending moments and transverse forces diagrams in a three-span statically determinate beam (Figure 2.13, a). The position of the design cross-sections on the beam is shown.

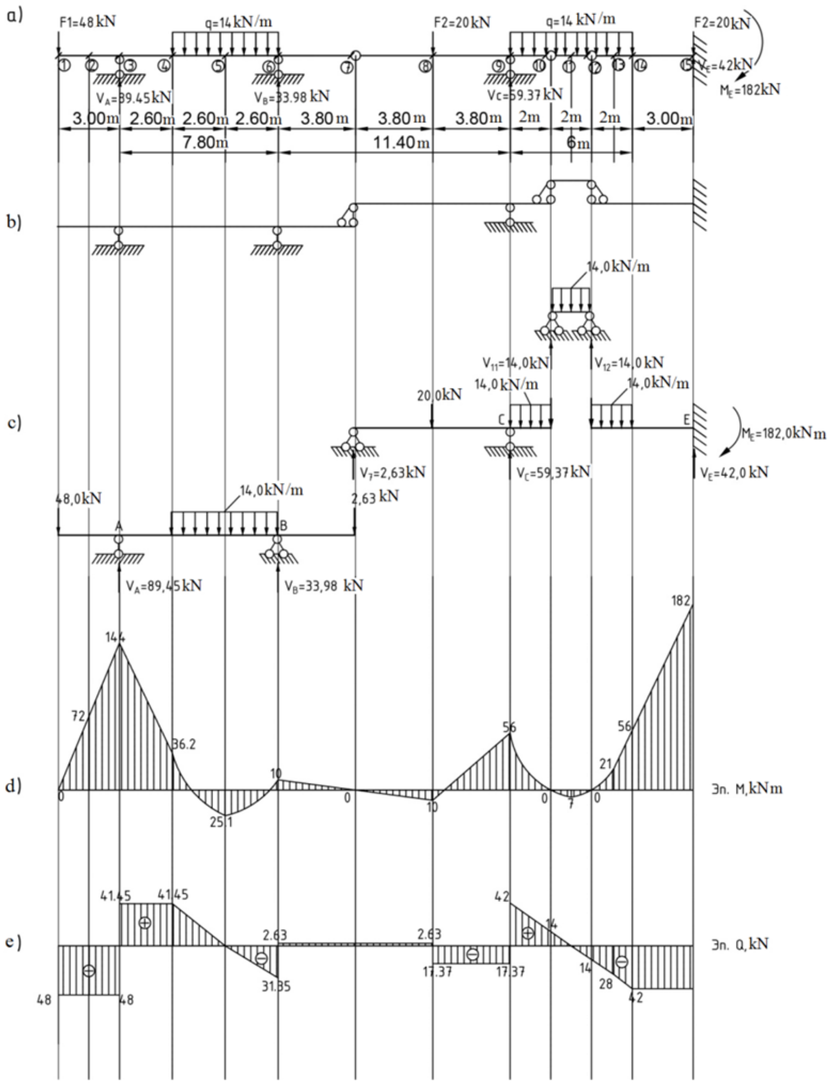


Figure 2.13

The degree of freedom of the beam is calculated by the formula:

$$W = 3D - 2H - S_0 = 3 \times 4 - 2 \times 3 - 6 = 0.$$

Breaking the beam by cross-sections 7, 10 and 12, we notice that the considered beam has two main parts: a simply supported beam AB (its length from section 1 to section 7 is 14.6 meters) and a cantilever beam (5 meters long from section 12 to a rigid fixed support at point E). The cantilever beam is rigidly fixed; there are three constraints at the right end of this beam. The horizontal beam AB is unmovable due to its binding (using non-deformable rods in the longitudinal direction at the segment 7–15) to rigid fixed support E. Considering that the beam (it can be called an insert) in the section 10–12 does not have its own support, we form the interaction scheme corresponding to figure 2.13, b.

Having determined the support reactions and the necessary efforts in the uppermost beam (on the interaction scheme), taking into account the interaction forces, it is necessary to transfer the pressure to the lower beams and continue their calculation. An illustration of the sequence of calculation of separate beams is on figure 2.13, c.

The internal forces diagrams for separate beams, which are being located horizontally in accordance with the position of the beams on a given scheme, form the internal forces diagrams for a multi-span beam (Figure 2.13, d, e).

E x a m p l e. For statically determinate compound frame (Figure 2.14, a) it is required to perform a kinematic analysis and to build the internal forces diagrams.

We perform kinematic analysis of the frame. Degree of freedom:

$$W = 3D - 2H - S_0 = 3 \times 3 - 2 \times 2 - 5 = 0.$$

We check the correctness of the frame structure and find its main and secondary parts. To do this we cut the design scheme (Figure 2.14, a) through the hinges which are connecting the disks, and analyze the mobility of each part. Having executed section only through the hinge K, we notice that each part of the frame (both left and right) is a geometrically changeable system. If we execute section only through the hinge F, then the left part of the frame will be geometrically unchangeable, unmovable: it will be a three-hinged frame with correctly located links (constraints). It will be the main part of the system.

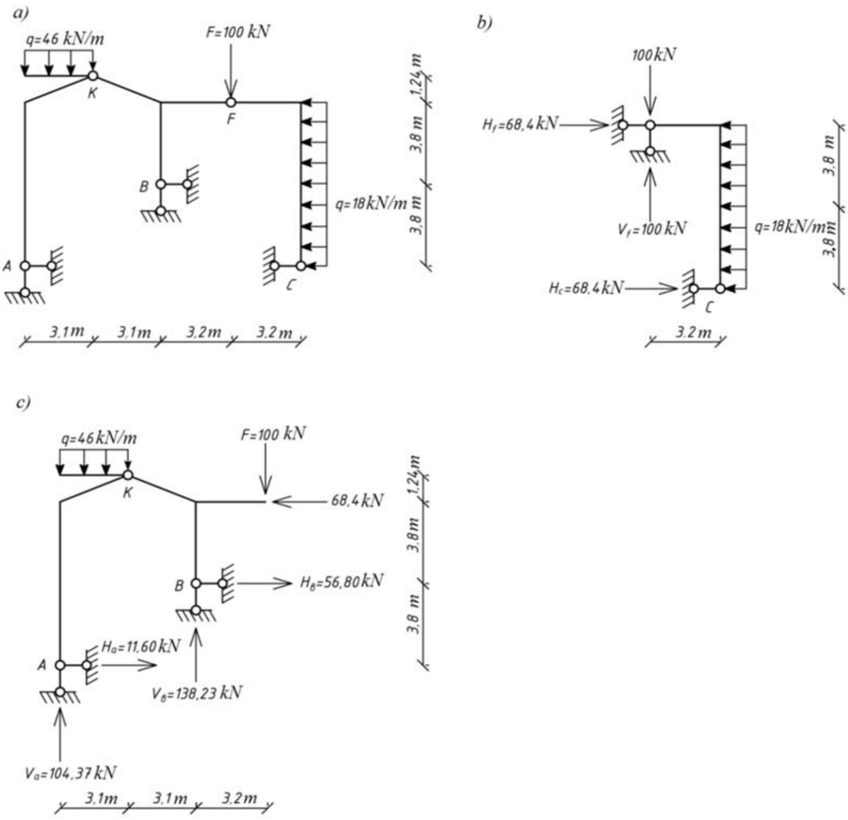


Figure 2.14

The right frame part will be also unchangeable, since it has its own support rod at point C, and at point F it is connected to the fixed frame by means of a hinge. The support rod at point C does not pass through the hinge F. The right part of the frame is auxiliary or secondary.

Then, a sequence of calculations is performed. It is characteristic of multi-span statically determinate beams.

Determining support reactions for the auxiliary frame:

$$\sum M_F = H_C \times 7.6 - 18 \times 7.6 \times 3.8 = 0;\ H_C = 68.4\,kN.$$

$$\sum Y = V_F - 100 = 0;\ V_F = 100\,kN.$$

Determining support reactions for the main frame:

$$\sum M_K^{right} = H_B \times 5.04 + V_B \times 3.1 - 68.4 \times 1.24 - 100 \times 6.3 = 0;$$
$$\sum M_A = V_B \times 6.2 - H_B \times 3.8 - 46 \times 3.1 \times 1.55 - 100 \times 9.4 + 68.4 \times 7.6 = 0;$$
$$H_B = 56.80\,kN;\ \ V_B = 138.23\,kN.$$

$$\sum M_K^{left} = H_A \times 8.84 - V_A \times 3.1 - 46 \times 3.1 \times 1.55 = 0;$$
$$\sum M_B = H_A \times 3.8 - V_A \times 6.2 + 46 \times 3.1 \times 4.65 - 100 \times 3.2 + 68.4 \times 3.8 = 0;$$
$$H_A = 11.60\,kN;\ V_A = 104.37\,kN.$$

Verifying the calculated support reactions for the main frame:

$$\sum X = H_A + H_B - 68.4 = 11.60 + 56.80 - 68.4 = 0;$$
$$\sum Y = V_A + V_B - 46 \times 3.1 - 100 = 104.37 + 138.23 - 46 \times 3.1 - 100 = 0.$$

Figure 2.15 shows the diagrams of bending moment (M), shear (Q) and longitudinal (N) forces.

Checking the balance of rigid nodes.

Figure 2.15, g shows the forces in the rods in sections adjacent to the node.

We compose the equilibrium equations of all forces (in this case, only internal) acting on the node.

$$\sum X = 0;\ 11.60 - 49.53 \times \cos\alpha + 92.60 \times \sin\alpha = 0;\ \cos\alpha = 0.9285;$$
$$\sin\alpha = 0.3719;$$

$$\sum Y = 0;\ 104.37 - 49.53 \times \sin\alpha - 92.60 \times \cos\alpha = 0.$$

We write the equilibrium equations of the forces shown in figure 2.15, h.

$$\sum X = 0;\ 56.80 - 68.40 - 31.19 \times \sin\alpha + 24.97 \times \cos\alpha = 0;$$
$$\sum Y = 0;\ 138.23 - 100 - 31.19 \times \cos\alpha - 24.97 \times \sin\alpha = 0;$$
$$\sum M_{node} = 0;\ 104.14 + 215.86 - 320 = 0.$$

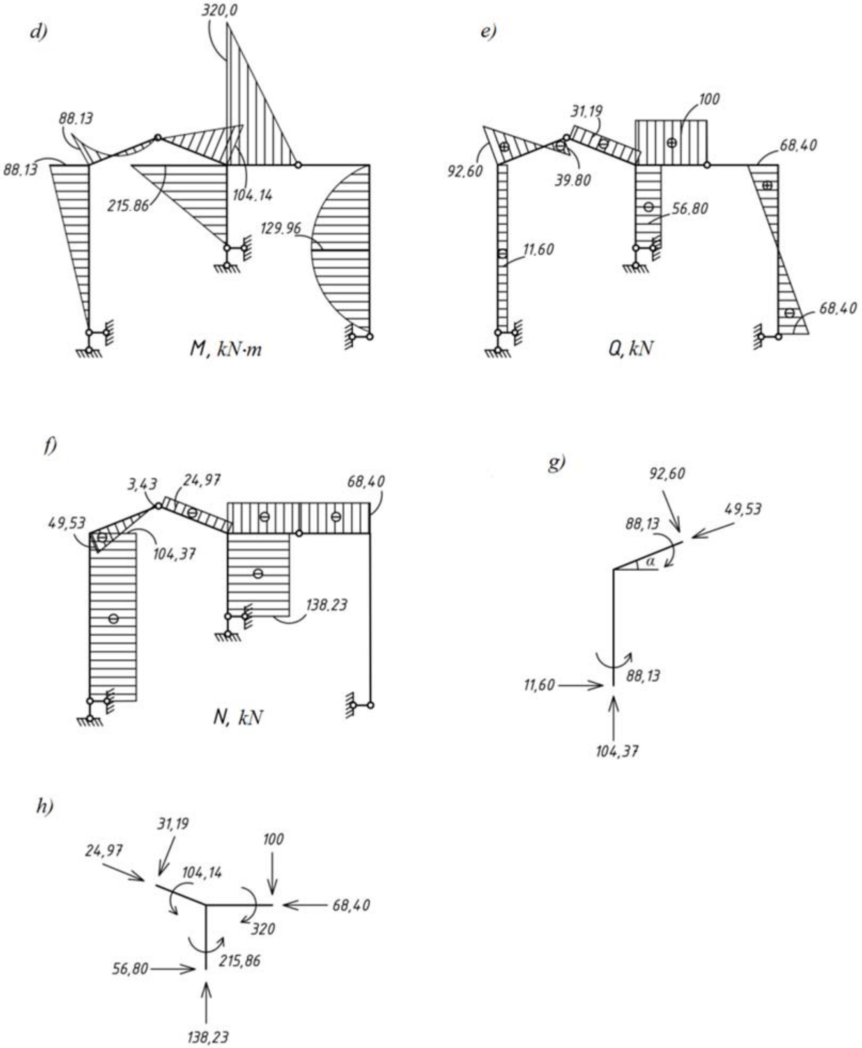


Figure 2.15

To check the balance of the frame as a whole, it is necessary to find support reactions and compose the required equilibrium equations. Practical actions are as follows: the frame elements are cut off from the supports; in the cross-sections of the elements the internal forces are shown, the numerical values of which are taken from the constructed diagrams; equilibrium equations are written in any of the previously listed forms.

In the considered example, after cutting the frame from the support (the picture is not shown), we are restricted by two equations:

$$\sum X = 0;\ -18\times 7.6+11.60+68.40+56.80=0;$$
$$\sum Y = 0;\ -4.6\times 3.1-100+104.37+138.23=0.$$

THEME 3. DETERMINATION OF EFFORTS FROM MOVING LOADS

3.1. Concept of Moving Load. Concept of Influence Lines

This theme discusses methods for calculating beam systems on the action of moving loads.

Moving are the loads that can move along the structure without changing the direction of action. A moving load is a load from automobile and railway transport, bridge cranes, etc. There is a wide variety of such loads. The pressure from such loads on the beam (or other structure) may be transmitted in the form of concentrated forces or may be distributed over some area (or length, in the case of plane systems).

To develop a general theory of calculation for all types of moving loads is a difficult task. The simplest elementary moving load is the concentrated unit force F = 1. Based on the knowledge about the influence of this force on any factor, it is possible to obtain a solution for any number of concentrated forces and loads distributed according to any law using the principle of independence of the forces action.

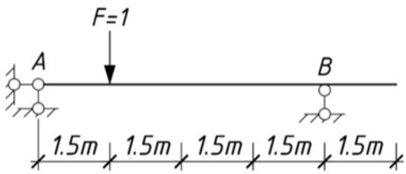


Figure 3.1

When the force F = 1 moves along the beam (Figure 3.1), the displacements of all its points are observed. For example, if the force is located at x = 1.5 m, then the displacements of the characteristic points of the beam (their coordinates are recorded in the left-hand column of Table 3.1) will be equal to the values indicated in the table for x = 1.5 m. Based on these values, you can construct the diagram of vertical displacements of the beam points. It is shown in Figure 3.2. The diagrams of the beam displacements at other positions of force can be constructed by corresponding values of the displacements of characteristic points using the data in Table 3.1.

Table 3.1

The coordinate of the point on the beam	The position of the force F = 1 on the beam			
	x=1.5 m	*x=3.0 m*	*x=4.5 m*	*x=7.5 m*
x=0.0 m	0	0	0	0
x=1.5 m	-2.53	-3.09	-1.97	2.11
x=3.0 m	-3.09	-4.50	-3.09	3.38
x=4.5 m	-1.97	-3.09	-2.53	2.95
x=6.0 m	0	0	0	0
x=7.5 m	2.11	3.38	2.95	-5.63

Note: 1. Apply a common factor 1/EI for all displacements.

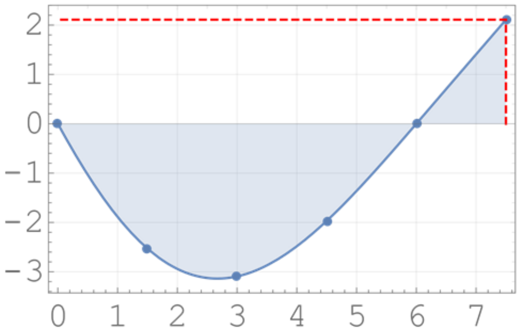


Figure 3.2. The diagram of vertical displacements of the beam points due to F = 1, located at the point x = 1.5 m

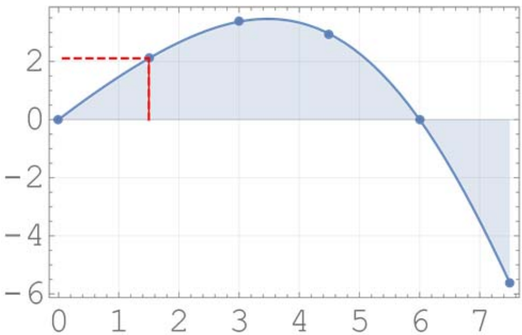


Figure 3.3. Influence line for the vertical displacement of one beam point with the coordinate x = 7.5 m

Using the displacement values in the last row of the table, we construct a displacements graph of the point at the end of the beam (x = 7.5 m) for all possible positions of the force F = 1 (Figure 3.3). Such a graph is called influence line for the displacement of the beam point with the coordinate x = 7.5 m. Carrying out similar considerations, it is possible to construct influence lines for internal forces (M, Q, N), which are stresses in some cross-sections of the beam, etc.

Definition. Influence line is a graph which shows **variation of some particular factor** (force, displacement, etc.) **in the given cross-section** of a structural element in terms of **position of unit** concentrated dimensionless **force** of a constant direction.

Note the differences in the concepts of "Influence line for an effort" and "Diagram of efforts".

The efforts diagram is a graph of some type efforts in all cross-sections of the structure loaded by fixed load. Influence line for the effort shows the effort in only one, fixed cross-section of the structure loaded by the moving force equal one.

3.2. Static Method of Constructing Influence Lines for Internal Forces

The previously described method of constructing influence lines requires a large number of beam calculations. The way in which the factor under investigation (in the previous example, displacement) is written as a function of the unit force position is more practical. This dependence can be obtained from the equations of equilibrium of a solid (equations of statics). The corresponding method of constructing influence lines is called static method.

3.2.1. Influence Lines for Support Reactions in a Simple Beam

We show construction of influence lines for efforts in a one-span beam (Figure 3.4 a).

We take the origin of the coordinate axes at point A. The X axis is directed along the axis of the beam, the Y axis is directed up. The position of the force F = 1 is determined by the x coordinate. On the Y axis we will plot the value of the investigated factor.

Writing the equation of the moments of all forces relative to point B, we obtain an expression that sets the dependence of the support reaction on the position of the force:

$$\sum M_B = 0; \quad V_A l - F(l-x) = 0;$$

$$V_A = \frac{F(l-x)}{l}. \tag{3.1}$$

Showing this relation graphically, we obtain the influence line for support reaction V_A (Figure 3.4, b).

Expression (1.1) is the equation of a straight line. To draw a line on a plane, it is enough to know the position of two points through which it passes. Find them, taking F = 1.

For $x = 0$ (the force is located above the support A) it follows from formula (3.1) that $V_A = 1$; for $x = l$ (the force is located above the support B) we obtain $V_A = 0$.

A straight line drawn through these two points represents the required influence line for support reaction (Figure 3.4, b).

In this example and in all subsequent ones positive ordinates of influence lines are drawn upward (in the direction of Y-axis).

We define the dimension of ordinates of the influence line for support reaction. If we take F = 1 in expression (1.1), then the right side of the equation can be written as follows:

$$\frac{(l-x)}{l}. \tag{3.2}$$

Comparing the record in the right-hand side of equation (3.1) and the right-hand side in the form (3.2) means dividing the left and right sides of equation (3.1) by F. In this case, equation (3.1) is transformed to

$$\frac{V_A}{F} = \frac{l-x}{l}. \tag{3.3}$$

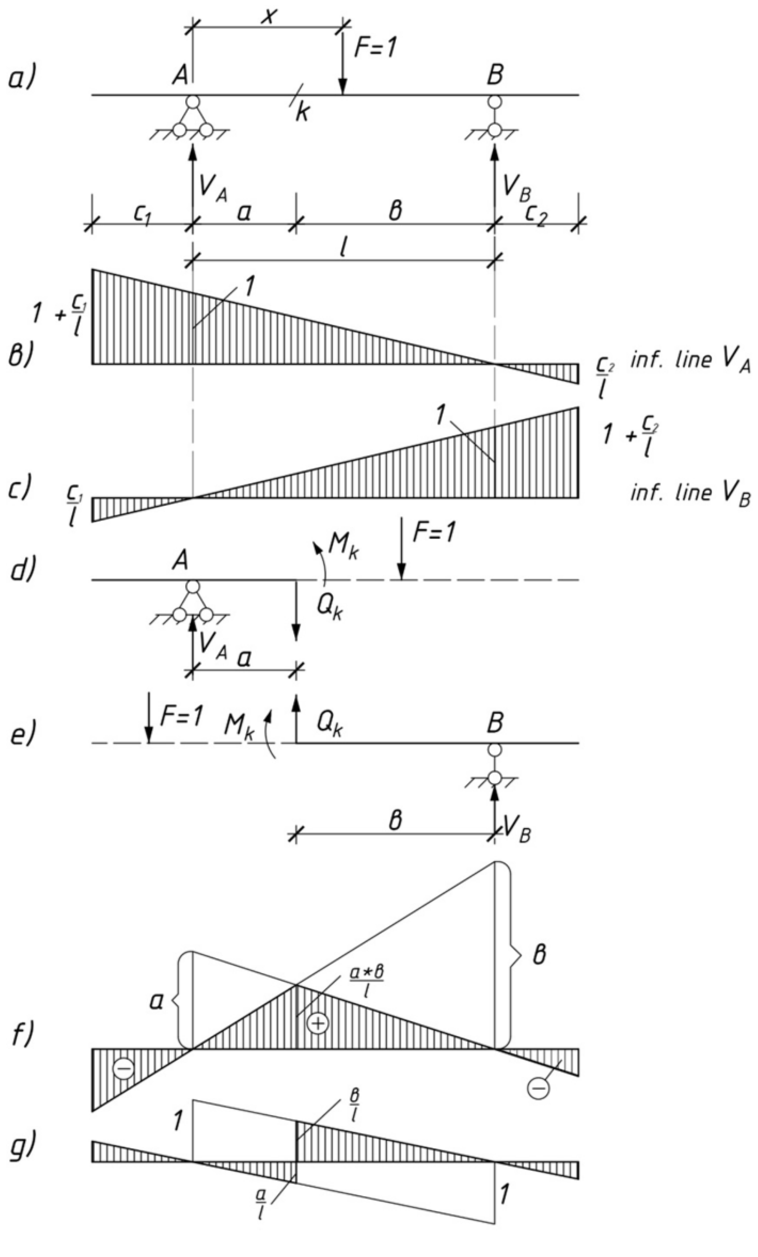

a)
x
F=1
A
B
k
V_A
V_B
c_1
a
в
c_2
l
1 + $\frac{c_1}{l}$
1
в)
$\frac{c_2}{l}$
inf. line V_A
1
1 + $\frac{c_2}{l}$
c)
$\frac{c_1}{l}$
inf. line V_B
F=1
M_k
d)
A
Q_k
V_A
a
F=1
M_k
Q_k
e)
B
в
V_B
в
a
$\frac{a*в}{l}$
f)
$\frac{в}{l}$
1
g)
$\frac{a}{l}$
1

Figure 3.4

Recording on the left of the equal sign indicates the dimension of the ordinates of the influence line for support reaction as a derivative of the dimensions of force factors. The dimension of the support reaction V_A and the force is F — kN. Consequently, the ordinates of influence line for support reaction have no dimension, they are dimensionless.

Analyzing these arguments in relation to the dimension of the ordinate, we obtain:

$$[\text{dimension of } \textit{the ordinate} \text{ of influence line for effort}] = \\ = \frac{[\text{dimension of the } \textit{required factor}]}{[\text{dimension of the } \textit{force}]}.$$

The unit ordinate at point A is the scale of the graph (a segment of any length is taken to be equal to one).

Writing the equation of the moments of all forces relative to point A, we obtain the expression for determining the support reaction V_B.

$$\sum M_A = 0; \quad V_B l - Fx = 0;$$

$$V_B = \frac{Fx}{l}. \tag{3.4}$$

To construct a line, we find the position of two points through which it passes. Taking F = 1, we get:

For $x = 0$ (the force is located above the support A) it follows from formula (3.4) that $V_B = 0$;

for $x = l$ (the force is located above the support B) we obtain $V_B = 1$.

The influence line for support reaction is shown in Figure 3.4, c.

3.2.2. Influence Lines for Efforts in Cross-Sections between Beam Supports

Design scheme of the beam is shown in Figure 3.4, a. The section k on the beam is fixed. Internal forces in section k of a beam depend on the position of a moving load F=1. The analytical dependences of the efforts in this section depend on the position of the force. It is located to the right-hand of section k or to the left-hand. Therefore, when determining the force in a cross-section, it is necessary to know where the force is located. The equilibrium equations are simpler, if when compiling them, we consider that part of the beam on which there is no force.

First, we construct influence lines for **bending moment** in the section ***k*.**

1. The force F = 1 is located to the right-hand of the section k ($a \le x \le l + c_2$).

From the equilibrium equations of the left side of the beam (Figure 3.4, d) it follows:

$$\sum M_k^{left} = 0;\ V_A a - M_k = 0;\ M_k = V_A a;\ V_A = \frac{l-x}{l};\ M_k = \frac{l-x}{l}a.$$

Influence line for M_k on the right side of the beam has the form of a straight line. We set for x the value from the interva ($a \le x \le l$):

$$x = a,\ M_k = \frac{l-a}{l}a = \frac{ab}{l};$$

$$x = l,\ M_k = 0.$$

The straight line constructed at these points is extended to the console, the length of which equals c_2 (Figure 3.4, f). Hatching (vertical) is performed on the **operating range (a** $\le$ ***x*** $\le$ $l + c_2$).

2. *The force F = 1 is located to the left-hand of the section* k ($-c_1 \le x \le a$).

From the equilibrium equations of the right side of the beam (Figure 3.4, d) it follows:

$$\sum M_k^{right} = 0;\ V_B b - M_k = 0;\ M_k = V_B b;\ V_B = \frac{x}{l};\ M_k = \frac{x}{l}b.$$

We construct a straight line.

$$x = 0,\ M_k = 0;$$

$$x = a,\ M_k = \frac{ab}{l}.$$

The straight line constructed at these points is extended to the console, the length of which equals c_1. (Figure 3.4, e). Hatching (vertical) is performed on the **operating range** ($-c_1$ $\leq$ **x** $\leq$ **a**).

[dimension of *the ordinate* of inf. line for bending moment] = $\dfrac{[kNm]}{[kN]} = m.$

Remark:

1. The formula $M_k = V_A a$ *can be read as follows:* inf. line M_k = = (inf. line V_A)a.

2. Analysis of the form of the inf. line M_k *shows that on the verticals passing through the support points, the inclined lines cut off segments equal to the distances from the supports to the section* k.

3. The top of the line of influence is located under the cross-section k.

We construct influence lines for **shear force** in the section ***k.***

1. The force F = 1 is located to the right-hand of the section k ($a \leq x \leq l$).

From the equilibrium equations of the left side of the beam (Figure 3.4, d) it follows:

$$\sum Y^{left} = 0;\ V_A - Q_k = 0;\ Q_k = V_A;\ Q_k = \frac{l-x}{l}.$$

Influence line for Q_k on the site of the position of the force can be constructed using inf. line V_A, or by the position of the points through which the line passes.

2. *The force F = 1 is located to the left-hand of the section* k ($0 \leq x \leq a$).

From the equilibrium equations of the right side of the beam (Figure 3.4, d) it follows:

$$\sum Y^{right} = 0;\ V_B + Q_k = 0;\ Q_k = -V_B;\ Q_k = -\frac{x}{l}.$$

Influence line for support reaction is shown in Figure 3.4, g.

[dimension of *the ordinate* of inf. line for shear force] = $\frac{[kN]}{[kN]}$ — ordinates are dimensionless.

3.2.3. Influence Lines for Efforts in the Cantilever Beam Sections

The design scheme of the beam is shown in Figure 3.5, a.

Construct influence lines for **bending moment** and **shear force** in the section ***k*.**

We take the origin of the coordinate axes in the section ***k***.

1. The force F = 1 is located to the right-hand of the section k ($0 \le x \le b$).

From the equilibrium equations of the right side of the beam (Figure 3.5, b) it follows:

$$\sum M_k^{right} = 0;\ M_k + Fx = 0;\ M_k = -Fx;\ \ M_k = -x.$$

$$\sum Y^{right} = 0;\ \ Q_k - F = 0;\ Q_k = F;\ \ Q_k = 1.$$

For $x = 0$ (the force is located in cross-section ***k***) $M_k = 0,\ Q_k = 1;$

при $x = b$ (the force is located above at the end of the console) $M_k = -l,\ Q_k = 1.$

2. *The force F = 1 is located to the left-hand of the section* k (–a ≤ x ≤ 0). **The right side** of the beam (Figure 3.5, c) is not loaded, therefore $M_k = 0,\ Q_k = 0.$

Influence lines for efforts is shown in Figures 3.5, d, e.

Let us once again draw attention to the interconnection of the concepts "influence line for effort" and "diagram of efforts". Figure 3.5 e shows the diagram of bending moments due to the force F = 1, appended at the end of the console. The ordinate on the diagram in cross-section k is equal to the ordinate of the influence line M_k at the end of the console (Figure 3.5, e).

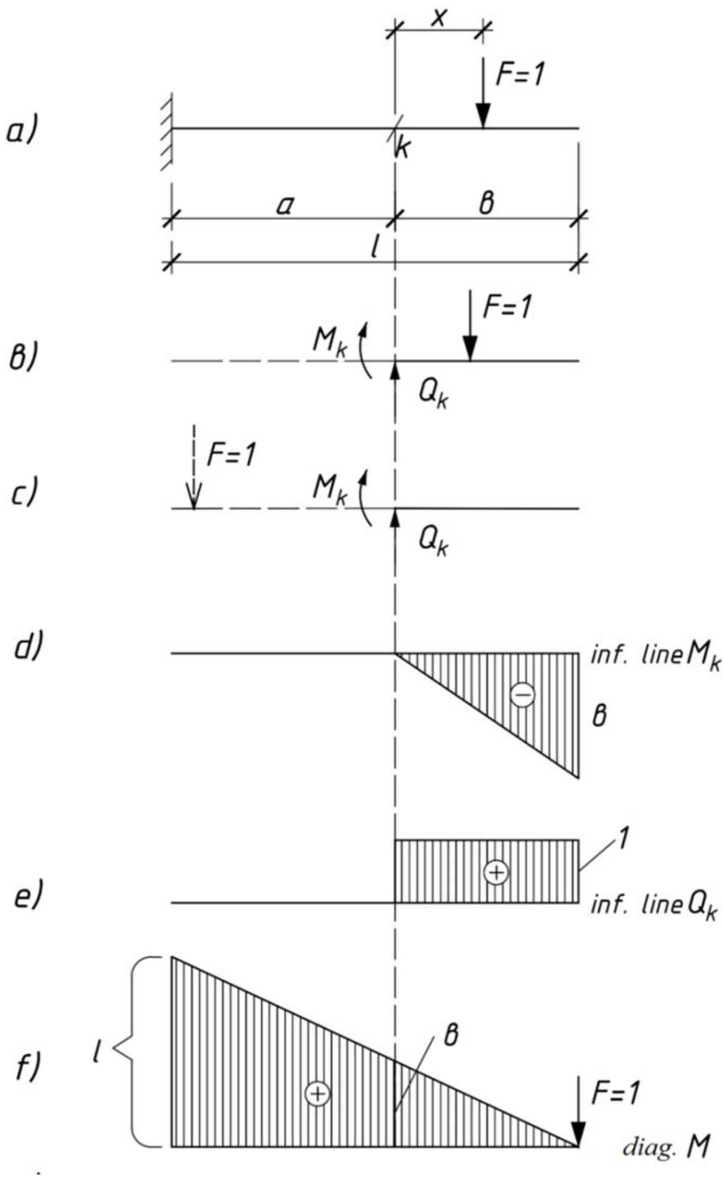

x
F=1
a)
k
a
в
l
F=1
в)
M_k
Q_k
F=1
c)
M_k
Q_k
d)
inf. line M_k
в
1
e)
inf. line Q_k
l
в
f)
F=1
diag. M

Figure 3.5

3.3. Kinematic Method for Constructing Influence Lines for Internal Forces

The kinematic method of constructing influence lines is based on the **principle of virtual displacements** (Section 2.4), according to which for a system that is in equilibrium under the action of external forces applied to it, the sum of the work of its external and internal forces on any infinitesimal displacements is zero.

Consider the design scheme of a simple beam (Figure 3.6, a).

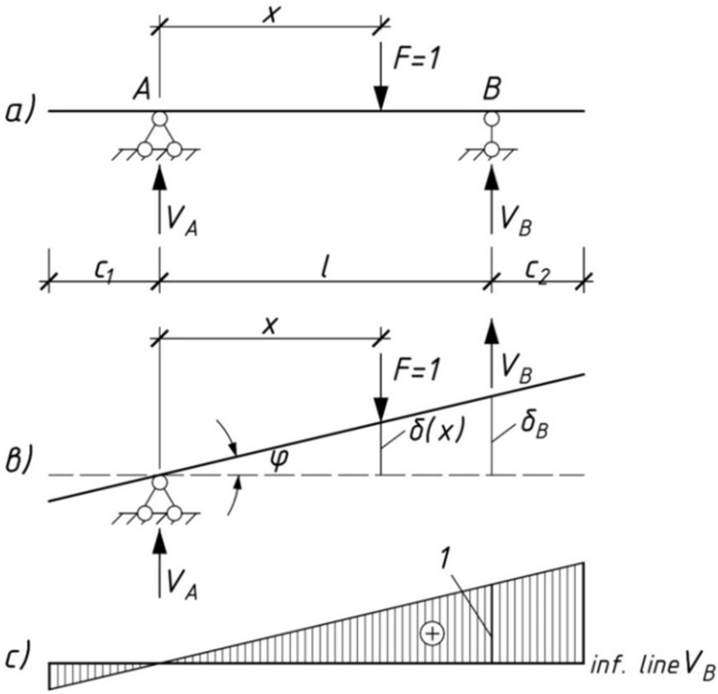


Figure 3.6

We construct influence line for support reaction V_B.

We eliminate the right support, replacing its action with a reaction V_B (Figure 3.6, b). The resulting system has become a mechanism. For the possible displacements take displacement caused by the rotation of the beam around the point A at an angle φ (Figure 3.6, b). We write down the sum of the forces acting on the system on the considered infinitesimal displacements:

$$-F\delta(x)+V_B\delta_B=0.$$

From this equation we get:

$$V_B=\frac{F\delta(x)}{\delta_B}. \tag{3.5}$$

Different positions of the force F = 1 lead to a change in the value of the corresponding displacement $\delta(x)$. In this case, all possible values of $\delta(x)$ along the length of the beam show a diagram of the vertical displacements of the beam points. The denominator in the formula (3.5) is a constant. δ_B is a scale factor. Assuming δ_B is equal to unity, we get:

$$V_B=\delta(x). \tag{3.6}$$

Consequently, the outline of the **influence line** coincides with the **diagram of the vertical displacements** of the points of the beam (Figure 3.6, c).

From the ordinate ratios in Figure 3.6,b we get $\frac{\delta(x)}{\delta_B}=\frac{x}{l}$, which, for F = 1, corresponds to the expression (3.4) obtained by the static method.

Construct influence lines for **bending moment** in the section *k*.

The design scheme of the beam is shown in Figure 3.7, a. We eliminate the constraint in the cross-section *k* through which the moment is transmitted (we set the hinge), replacing its action with the moment M_K (Figure 3.7, b). The figure shows the interaction forces of the left and right parts of the beam. We will set the possible displacements to the obtained mechanism in the direction of the moments M_K action, taking the angle of mutual rotation of the end cross-sections equal to unity. The ordinates between the initial position of the beam and the new (broken) form a diagram of the beam displacement (Figure 3.7, b).

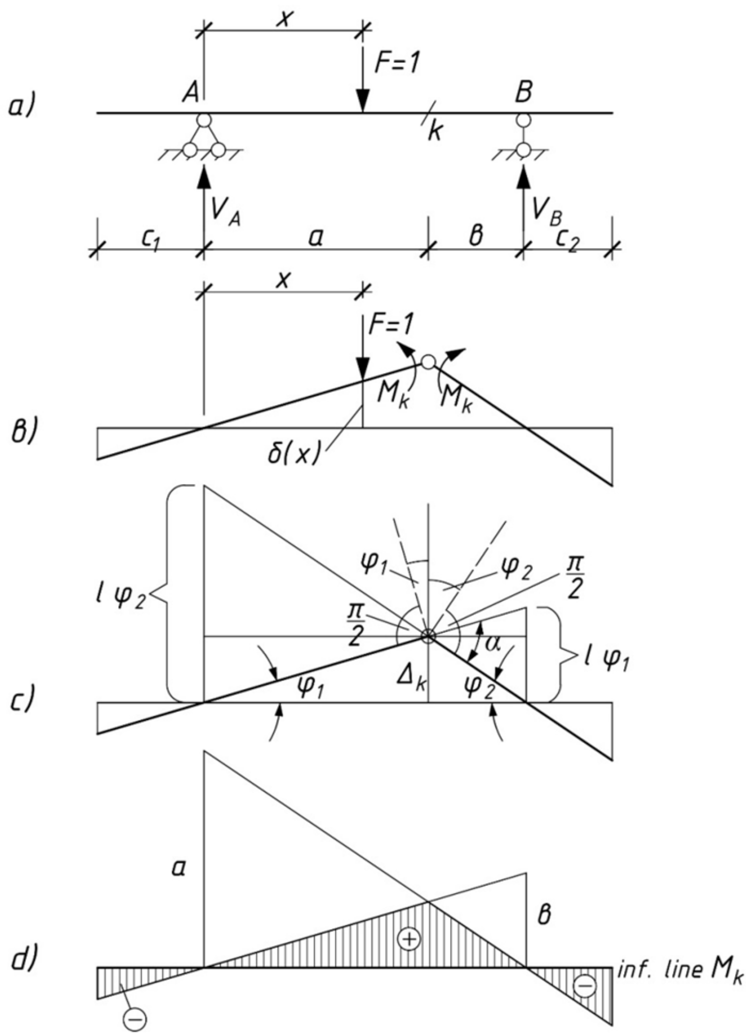


Figure 3.7

The virtual work of external and internal forces on the taken beam displacements is equal to zero:

$$-F\delta(x) + M_K 1 = 0.$$

At F = 1 we get $M_K = \delta(x)$, that corresponds to the above conclusion: influence line M_K (Figure 3.7, d) coincides with the diagram of the vertical displacements of the beam points.

Using the notation given in Figure 3.7,a, shows that it is exactly coincides with the influence line previously constructed by a static method (Figure 3.4, e).

Possible displacements, in fact, are infinitesimal. Therefore, when analyzing the relations in Figure 3.7, you can use simplifications of the form:

$$\mathrm{tg}\varphi_1 \approx \varphi_1; \;\; \mathrm{tg}\varphi_2 \approx \varphi_2.$$

From the data in Figure 3.7, c, provided that $\alpha = \varphi_1 + \varphi_2 = 1$, we obtain:

$$l\varphi_1 = b\varphi_2 + b\varphi_1; \;\; l\,\varphi_1 = b; \;\; \varphi_1 = \frac{b}{l}; \;\; \Delta_k = a\varphi_1 = \frac{ab}{l}.$$

$$l\varphi_2 = a\varphi_1 + a\varphi_2; \;\; l\,\varphi_2 = a; \;\; \varphi_2 = \frac{a}{l}; \;\; \Delta_k = b\varphi_2 = \frac{ab}{l}.$$

The ordinate of influence line in cross-section k is equal to the ordinate obtained by the static method (Figure 3.4, f).

Let us construct the influence lines for **shear force** in the section k (Figure 3. 8, a).

We eliminate the constraint in this cross-section, in which a shear force can arise. The connection of the left and right parts of the beam after this is carried out by means of two horizontally arranged links through which longitudinal forces and bending moments can be transmitted. On the newly formed design scheme, we show in the cross-section the positive directions of the shear forces for both parts of the beam (Figure 3.8, b). Giving the unity value for mutual displacement of the beam ends along the directions of the shear forces, we obtain a diagram of the beam's displacements (Figure 3.8, c), the outline of which completely corresponds to influence line for shear force (Figure 3.8, d).

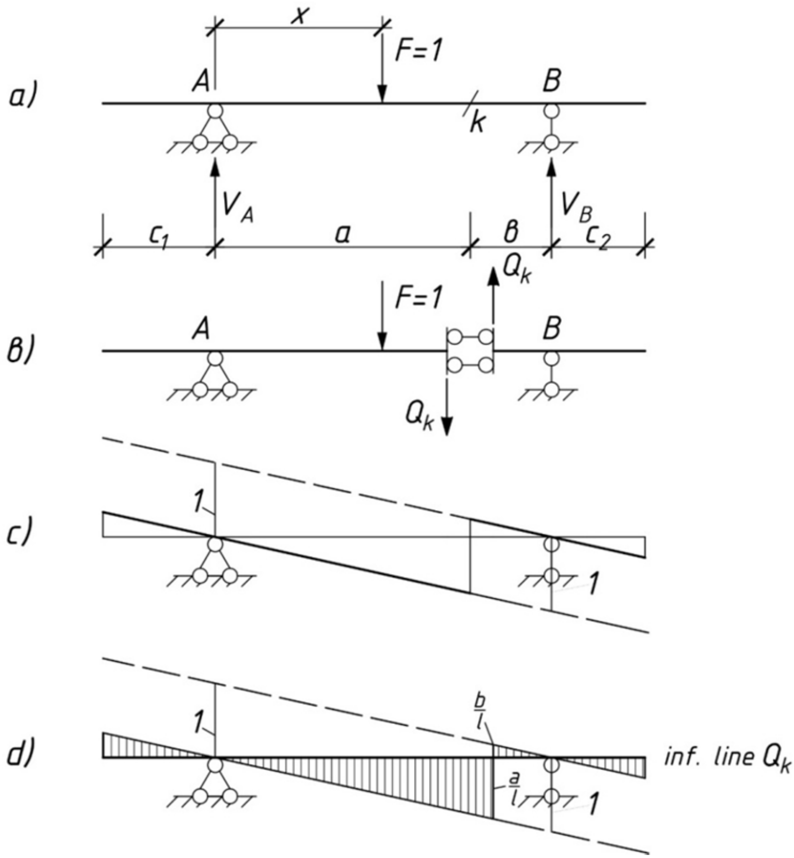


Figure 3.8

3.4. Determination of the Effort from Fixed Load Using Influence Lines

By the definition, each of the ordinates of the inf. line for S represents the value of the effort S when the acting force F = 1 is located on the beam above this ordinate. If a unit force is not located above the ordinate, but a force whose value is equal F is located there, then the effort caused by its action will be F times more, i.e. the effort will be equal to the product of the force F and the ordinate of the influence line for the effort under this force: $S = Fy$.

If n **concentrated vertical** forces act on the beam (Figure 3.9), then the force S, based on the principle of superposition, should be calculated by the formula:

$$S = F_1 y_1 + F_2 y_2 + ... + F_n y_n = \sum_{i=1}^{i=n} F_i y_i. \tag{3.7}$$

In this expression, the value of the looking downward force is taken with the plus sign, the value of the looking upward force is taken with the minus sign.

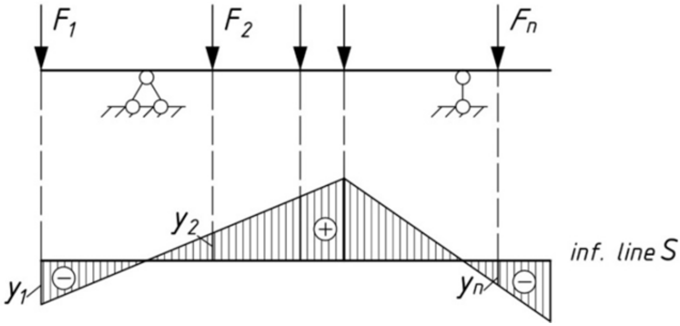


Figure 3.9

Consider the action on the beam of a **load distributed** according to an arbitrary law $q(x)$, (Figure 3.10, a). On this beam, we select a section of infinitely small length dx. The concentrated force replacing the distributed load on this section is equal to $dF = q(x)dx$. (Figure 3.10, a). The elementary effort dS from the action of the force dF is:

$$dS = dFy = q(x)y(x)dx.$$

Integrating this expression along the length of the loading section, we find:

$$S = \int_a^b q(x)y(x)dx. \tag{3.8}$$

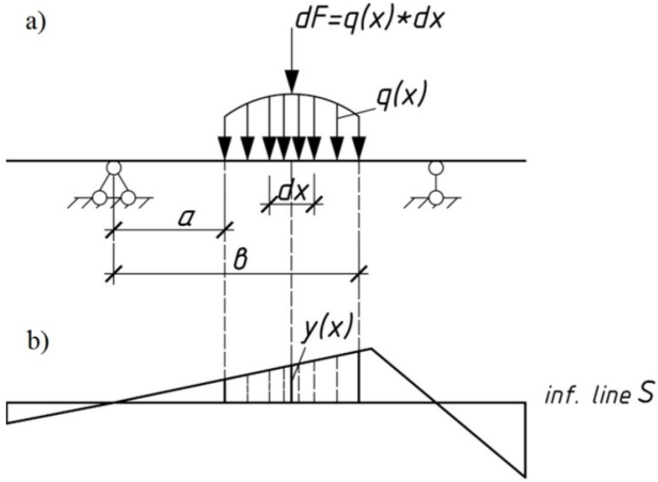


Figure 3.10

If a **uniformly distributed load** acts on the beam $q(x) = q$ (Figure 3.10, b), then

$$S = \int_a^b q y(x) dx = q \int_a^b y(x) dx = q\omega. \tag{3.9}$$

Here ω is the area of influence line S corresponding the uniformly distributed load action site. In figure 3.10, b the area ω is highlighted by hatching. It should be kept in mind that the ordinates of the influence lines located above the axis of the beam are positive, the ordinates of the lines of influence located below the axis of the beam are negative. The area below the axis is negative.

Let us consider the action on the beam of a **concentrated moment** M. (Figure 3.11). Replace the moment with a couple of forces F with arm Δx: $F = \dfrac{M}{\Delta x}$. With the help of the formula (3.7) we find:

$$\begin{aligned} S &= \lim_{\Delta x -> 0} \left[-\frac{M}{\Delta x} y + \frac{M}{\Delta x}(y + \Delta y) \right] = M \lim_{\Delta x -> 0} \left[\frac{y + \Delta y}{\Delta x} - \frac{y}{\Delta x} \right] = \\ &= M \lim_{\Delta x -> 0} \frac{\Delta y}{\Delta x} = M \frac{dy}{dx}. \end{aligned} \tag{3.10}$$

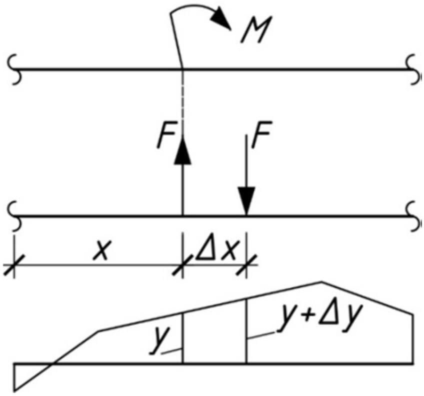


Figure 3.11

The moment directed in a clockwise direction, is considered positive. The value of the derivative of the function that describes the outline of influence line is calculated at the point of application of the concentrated moment.

On a straight section of the influence line, the calculation of the effort S will be a simpler action if the concentrated moment is replaced by a pair of forces on any length of this section.

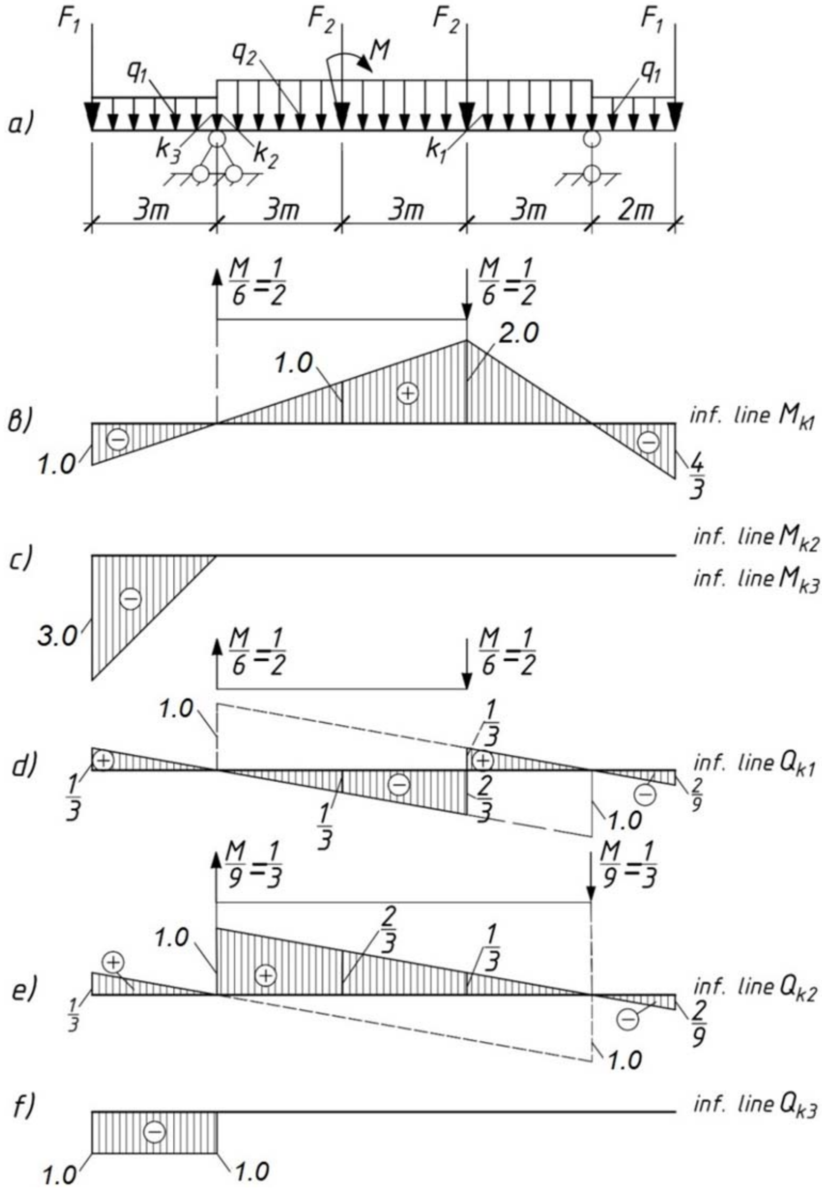
With the simultaneous action on the beam of all considered force factors (concentrated forces, distributed load, concentrated moment), the effort S is calculated by summing the results caused by each factor individually based on the principle of superposition.

E x a m p l e. Using influence lines to determine bending moments and shear forces in sections k_1, k_2 and k_3 of the beam (Figure 3.12, a) with the following data:

$$F_1 = 4kN;\; F_2 = 10kN;\; q_1 = 2kN/m;\; q_2 = 5kN/m;\; M = 3kNm.$$

The cross-section k_2 is infinitely close to the support A on the right-hand, the cross-section k_3 is infinitely close to the support A on the l eft-hand.

The influence lines for efforts are shown in Figures 3.12, b ... 3.12, e.



F1
F2
F2
F1
q1
q2
M
q1
a)
k3
k2
k1
3m
3m
3m
3m
2m
M/6 = 1/2
M/6 = 1/2
2.0
1.0
в)
1.0
4/3
inf. line Mk1
c)
inf. line Mk2
inf. line Mk3
3.0
M/6 = 1/2
M/6 = 1/2
1.0
1/3
d)
1/3
1/3
2/3
1.0
2/9
inf. line Qk1
M/9 = 1/3
M/9 = 1/3
1.0
2/3
1/3
e)
1/3
2/9
1.0
inf. line Qk2
f)
inf. line Qk3
1.0
1.0

Figure 3.12

We find the efforts:

$$M_{k_1} = 4\cdot(-1)+4\cdot(-4/3)+10\cdot 1+10\cdot 2+2\cdot(-1/2\cdot 1\cdot 3)+$$
$$+2\cdot(-1/2\cdot 2\cdot 4/3)+5\cdot 1/2\cdot 2\cdot 9-1/2\cdot 0+1/2\cdot 2=61kNm;$$

$$M_{k_2} = M_{k_3} = 4\cdot(-3)+2\cdot(-1/2\cdot 3\cdot 3) = -21kNm;$$

$$Q_{k_1}^{left} = 4\cdot 1/3+4\cdot(-2/9)+10\cdot(-1/3)+10\cdot 1/3+2\cdot(1/2\cdot 1/3\cdot 3)+$$
$$+2\cdot(-1/2\cdot 2/9\cdot 2)+5\cdot(-1/2\cdot 2/3\cdot 6+1/2\cdot 1/3\times 3)-$$
$$-1/2\cdot 0+1/2\cdot(-2/3) = -6.833kN;$$

$$Q_{k_1}^{right} = 4\cdot 1/3+4\cdot(-2/9)+10\cdot(-1/3)+10\cdot(-2/3)+2\cdot(1/2\cdot 1/3\cdot 3)+$$
$$+2\cdot(-1/2\cdot 2/9\cdot 2)+5\cdot(-1/2\cdot 2/3\cdot 6+1/2\cdot 1/3\cdot 3)-$$
$$-1/2\cdot 0+1/2\cdot(-2/3) = -16.833kN;$$

$$Q_{k_2} = 4\cdot 1/3+4\cdot(-2/9)+10\cdot 2/3+10\cdot 1/3+2\cdot(1/2\cdot 1/3\cdot 3)+$$
$$+2\cdot(-1/2\cdot 2/9\cdot 2)+5\cdot(1/2\cdot 1\cdot 9)-1/3\cdot 1+1/3\cdot 0 = 33.167kN;$$

$$Q_{k_3} = -4\cdot 1+2(-1\cdot 3) = -10kN.$$

Note. *Other factors can be defined similarly if the corresponding influence lines are constructed for them.*

Let us turn to Figure 3.3, which shows influence line of the vertical displacement of the beam point with a coordinate $x = 7.5m$.

Using the displacements given in Table 3.1 for characteristic points, we find an approximating polynomial that describes the outline of influence line and the first derivative of it:

$$p(x) = [1.59222x - 0.130259x^2 + 0.0190617x^3 - 0.0115391x^4 +$$
$$+ 0.000768176x^5]\frac{1}{EI}.$$

$$\frac{dp(x)}{dx} = [1.59222 - 0.260519x + 0.0571852x^2 - 0.0461564x^3 +$$
$$+ 0.00384088x^4]\frac{1}{EI}.$$

Consider loading a beam with a uniformly distributed load and a concentrated moment at a point $x = 7.5m$ (Figure 3.13).

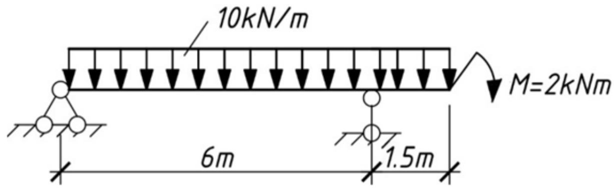


Figure 3.13

Find the displacement, knowing the outline of influence line and the first derivative:

$$Z_{x=7.5}^{vert} = q\omega + M\frac{dy}{dx} = q\int_{0}^{7.5} p(x)dx + M\frac{dp(x)}{dx}.$$

$$\int_{0}^{7.5} p(x)dx = 9.5625\frac{1}{EI}; \quad \left.\frac{dp(x)}{dx}\right|_{x=7.5} = -4.46444\frac{1}{EI}.$$

$$Z_{x=7.5}^{vert} = 10\cdot\frac{9.5625}{EI} - 2\cdot\frac{4.46444}{EI} = \frac{86.6961}{EI}.$$

3.5. Influence Lines for Efforts in Case of the Nodal Transfer of the Load

Consider the construction design scheme shown in Figure 3.14, a. The main bearing element of this scheme is the beam AB. It is called the main beam. The main beam bears cross beams. They are presented on the design scheme in the form of support rods for short longitudinal beams located at the upper level. Short beams are essentially flooring performed in the simplest case of planks. The load (force F = 1 is shown on the design scheme) applied to the upper short beams is transferred to the main beam at specific points, which are called nodes.

Hence the name follows: nodal transfer of the load.

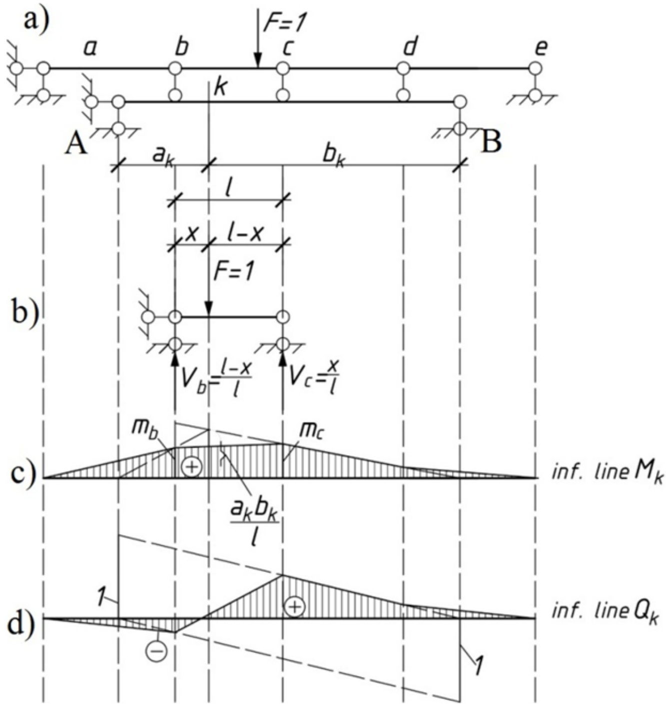


Figure 3.14

Nodal transfer of the load is used frequently in constructions. This takes place in arches with a superstructure, when transferring the load to the nodes of the trusses through the ribbed slabs of the roof (or floor) and in other cases.

We show features of influence lines construction in case of nodal transfer of the load. Firstly, we construct influence line for bending moment in the cross-section k under the assumption that the superstructure above the main beam is absent and the force moves directly upon the main beam (Figure 3.14, c).

The force F = 1 located on the beam *bc* (Figure 3.14, b) causes the reactions

$$V_b = \frac{l-x}{l} \quad and \quad V_c = \frac{x}{l}$$

Considering them as the forces of interaction between the beam *bc* and the beam *AB*, we obtain the loading of the beam *AB*. By formula 3.7 we find the moment in the cross-section k:

$$M_k = \frac{l-x}{l} m_b + \frac{x}{l} m_c.$$

The equation of the line passing through the points:

$$x = 0 \;\; M_k = m_{b.}; \quad x = l \;\; M_k = m_c,$$

is obtained.

Consequently, the location of the force F = 1 on the beam *bc* corresponds to a straight line (it is also called a transfer line) passing through the tops with the ordinates $x = 0$ and $x = l$ of the previously constructed influence line M_k. A similar result will be obtained when the force moves upon the other beams of the upper structure: on the section of each beam, influence line for effort will be straight.

So, to construct the influence line for an effort *S* with the nodal transfer of the load, you must:

– construct the influence line for an effort *S* as if the moving unit load would be applied directly to the main beam.

– transfer the nodes on the constructed influence line *S* and obtain the ordinates on it;

– connect the tops of the ordinates with straight lines.

Figures 3.14, c, d show the influence lines for M_k and Q_k.

3.6. Construction of the Influence Lines for Efforts in Multi-Span Beams

With the known interactive scheme of a multi-span beam, the construction of influence line for effort *S* starts with the beam to which analyzed factor belongs. Plotting is performed by the static or kinematic method. Having received the influence line for this beam, we should continue the construction for the adjacent upward beam, that is, we should consider the position of the force F = 1 on it. The ordinate of influence line in the hinge connecting the lower and upper beams is the same. The second ordinate on the upper beam is equal to zero and is located above the support of this beam, since the force F = 1 is above the support, the effort $S = 0$. Having two known ordinates, we show the position of the line along the entire length of the beam. The process of constructing is repeated for all upward beams.

Figure 3.15 shows the influence lines for the efforts in a multi-span beam.

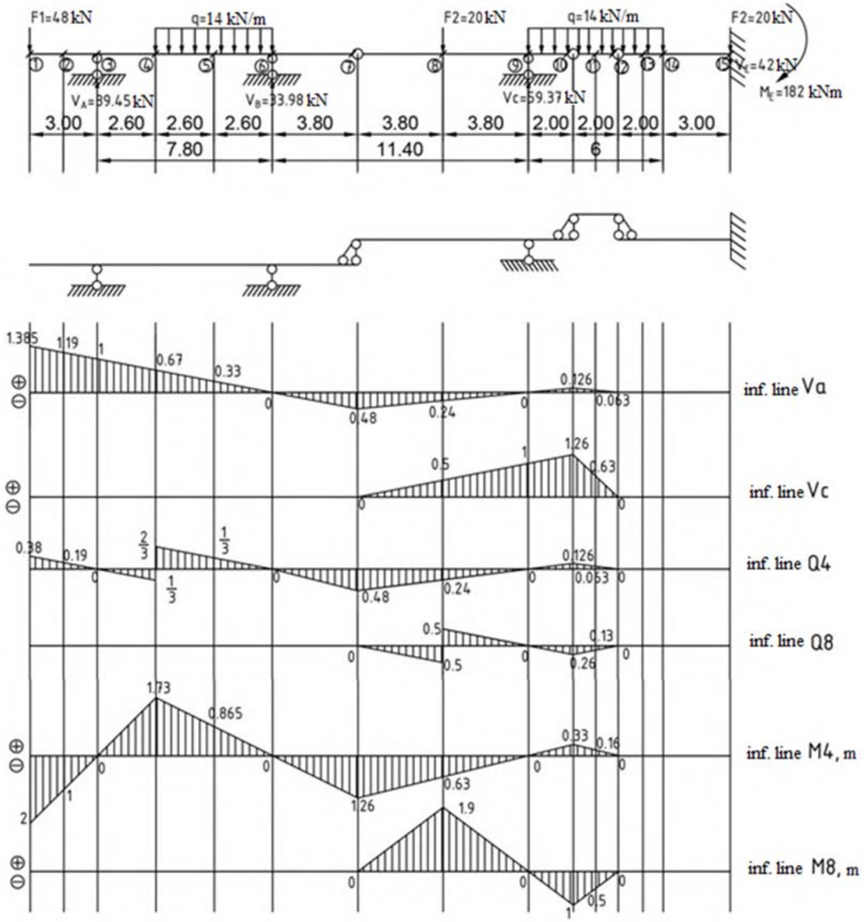


Figure 3.15

3.7. Determining the Most Unfavorable Position of Moving Loads with Influence Lines

The most unfavorable position of a moving load upon the structure is the position in which the considered effort reaches its maximum (extreme) value.

3.7.1. Concentrated force action

Consider the case when there is **one single concentrated force F** on the beam (Figure 3.16). Influence line for the effort S is built. For any position of the force on the beam, the effort S will be calculated by the formula (3.7): $S = Fy$. The effort will be maximum if the force $F = \text{const}$ is located above the maximum ordinate of influence: $S_{\max} = Fy_{\max}$. It is clear that $S_{\min} = Fy_{\min}$.

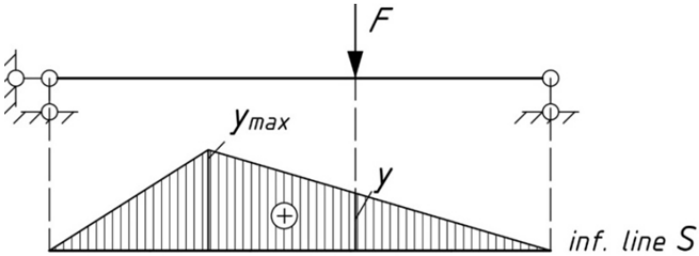


Figure 3.16

3.7.2. Action of a Set of Connected Concentrated Loads

The set of connected moving loads, shown in figure 3.17, simulates the pressure of train wheels or other transport. The distance between the forces does not change when the train moves. All forces are located on a certain section of the triangular influence line (Figure 3.17, b). The force F_i is located on the left, at a very small distance from the vertex of the influence line.

The effort S from the shown load is calculated by the formula (3.7):

$$S = F_1 y_1 + F_2 y_2 + \ldots + F_i y_i + \ldots + F_n y_n.$$

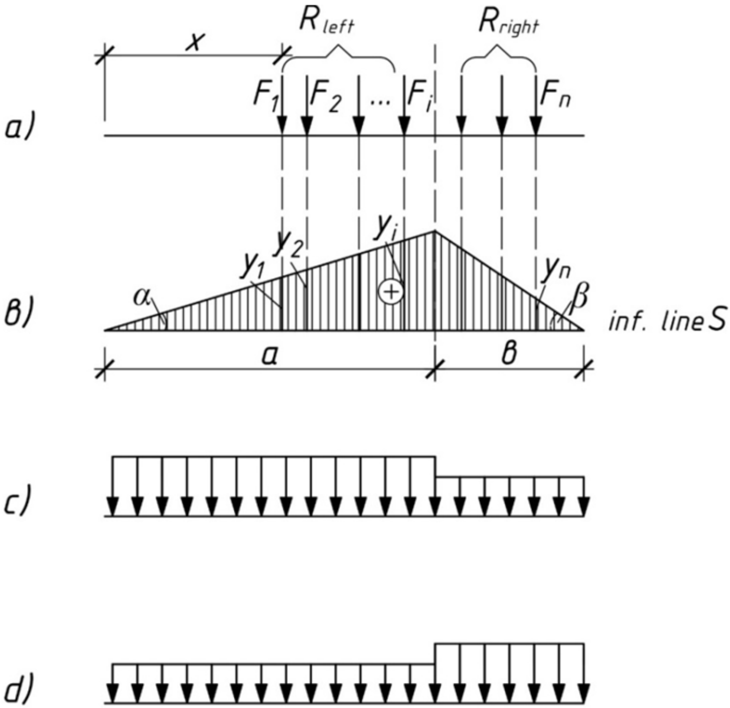


Figure 3.17

When the train moves, all ordinates $y = y(x)$ are variable. Consequently, the effort $S = S(x)$ is also variable. We are looking for the extremum of the function $S(x)$.

The first derivative of S has the form:

$$\frac{dS}{dx} = F_1\frac{dy_1}{dx} + F_2\frac{dy_2}{dx} + \ldots + F_i\frac{dy_i}{dx} + \ldots + F_n\frac{dy_n}{dx} = (R_{left} + F_i)\text{tg}\alpha - R_{right}\text{tg}\beta,$$

where $\quad \dfrac{dy_1}{dx} = \dfrac{dy_2}{dx} = \ldots = \dfrac{dy_i}{dx} = \text{tg}\alpha, \quad \dfrac{dy_n}{dx} = \text{tg}(\pi - \beta) = -\text{tg}\beta.$

R_{left} – is the resultant of forces located to the left of the force F_i (on influence line of length *a*);

R_{right} – is the resultant of forces located to the right of the vertex of the influence line**.**

The function $S(x)$ is not smooth, when the force F_i is transferred to a portion of the right branch of influence line, the first derivative $\frac{dy_i}{dx}$ changes sign from "plus" to "minus" in form of a break of the first kind. Therefore, you cannot use equality $\frac{dS}{dx}=0$ to calculate the extreme value S.

A note on the change of the first derivative sign means that the extreme value S will be observed when one of the concentrated forces is located above the top of the influence line. Suppose this happens when a force F_i is located above the vertex of the influence line. Then this force is called critical and is denoted as follows: $F_{кр} = F_i$.

The condition for determining the critical force is written in the form of two inequalities:

$$\begin{aligned}&(R_{left}+F_{cr})\text{tg}\alpha \geq R_{right}\text{tg}\beta;\\&R_{left}\text{tg}\alpha \leq (R_{right}+F_{cr})\text{tg}\beta.\end{aligned} \tag{3.11}$$

If both inequalities are satisfied simultaneously, then F_i is a critical force, and the corresponding load position is called the unfavorable one (estimated). If inequalities are not satisfied at the same time, then we must assume that another force will be critical and verify that the criterion (3.11) is satisfied.

Inequalities (3.11) can be given a graphical interpretation. It is given that $\text{tg}\alpha=\frac{c}{a}$, $\text{tg}\beta=\frac{c}{b}$, inequalities show the ratio of equivalent uniformly distributed loads on the left-hand and right-hand sections of influence line (Figure 3.17, d).

The action of **two related forces** (Figure 3.18) can be regarded as a special case of the considered load case. In all the loads considered in the example, the movement of the load from right to left is received.

For the first loading (Figure 3.18, b) $S_{\max}^{(1)} = F_1 y_1 + F_2 y_2$; for the second loading (Figure 3.18, c) $S_{\max}^{(2)} = F_1 y_3 + F_2 y_1$.

From the found values of the efforts, we select the larger one $S_{\max} = \max\{S_{\max}^{(1)}, S_{\max}^{(2)}\}$ and obtain information of the position of the load is the unfavorable one and its force is critical.

For the third loading (Figure 3.18, d) $S_{\min}^{(3)} = F_1 y_4 + F_2 y_5$; for the fourth loading – $S_{\min}^{(4)} = F_1 y_5$, if the position of force F_2 outside the beam is possible.

Further, from the found values of the efforts, we choose the smaller one $S_{\min} = \min\{S_{\min}^{(3)}, S_{\min}^{(4)}\}$. Then, from the found values of the efforts, we choose the smaller one. The position of the load at which the effort will be minimal is the unfavorable.

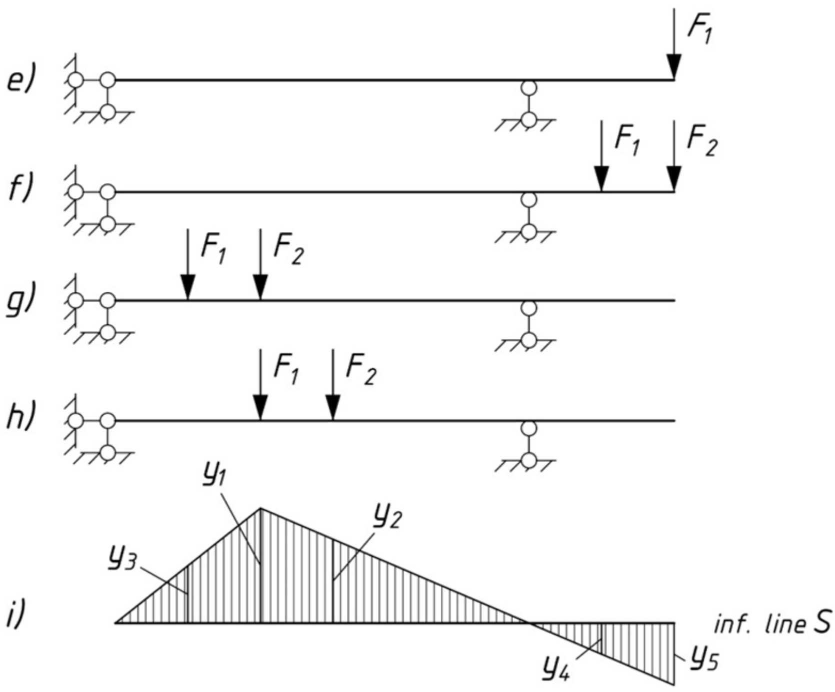


Figure 3.18

3.8. Influence Matrices for Internal Forces

We define an effort S_k in the cross-section k of the beam (Figure 3.19) caused by the concentrated forces F_i $(i = 1,...,n)$ applied to that beam. For a linearly deformable system, any internal force S_k in the cross-section k, $(k = \overline{1,m})$ can be determined by the expression:

$$S_k = s_{k1}F_1 + s_{k2}F_2 + ... + s_{kn}F_n, \tag{3.12}$$

where s_{ki} – is the effort in cross-section k due to $F_i = 1$.

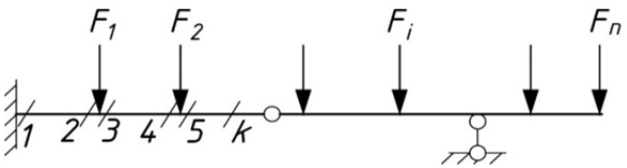


Figure 3.19

We represent expression (3.12) in the expanded form for $k = \overline{1,m}$.

$$\begin{aligned} S_1 &= s_{11}F_1 + s_{12}F_2 + ... + s_{1n}F_n; \\ S_2 &= s_{21}F_1 + s_{22}F_2 + ... + s_{2n}F_n; \\ &... \\ S_m &= s_{m1}F_1 + s_{m2}F_2 + ... + s_{mn}F_n; \end{aligned} \tag{3.13}$$

In matrix form, the system of equations (3.13) has the following form:

$$\vec{S} = L_S\vec{F}. \tag{3.14}$$

Here $\vec{S}$ is a vector of effort; $\vec{F}$ – a load vector; L_S – an influence matrix for the efforts $\vec{S}$:

$$\vec{S} = \begin{bmatrix} S_1 \\ S_2 \\ ... \\ S_m \end{bmatrix}; \quad \vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ ... \\ F_n \end{bmatrix}; \quad L_S = \begin{bmatrix} s_{11} & s_{12} & ... & s_{1n} \\ s_{21} & s_{22} & ... & s_{2n} \\ ... & & & \\ s_{m1} & s_{m2} & ... & s_{mn} \end{bmatrix}. \tag{3.15}$$

Influence matrix L_S is a linear operator that transforms the load vector into the efforts vector.

If bending moments are determined, then the matrix L_S is denoted L_M and is called the influence matrix of bending moments. In this case, equations (3.14) are written in the form:

$$\vec{M} = L_M \vec{F}, \tag{3.16}$$

where $\vec{M}$ – is a vector of bending moments in the calculated sections, and the matrix is written as follows:

$$L_M = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \dots & & & \\ m_{m1} & m_{m2} & \dots & m_{mn} \end{bmatrix}. \tag{3.17}$$

In the general case, this matrix is rectangular; its dimension is $(m \times n)$. In the case when concentrated forces are applied in the calculated sections, the matrix L_M is a square matrix of order n

Since m_{ki} is the bending moment in the cross-section k caused by the force $F_i = 1$, then, analyzing the matrix L_M, we notice that in each of its row the ordinates of the corresponding influence lines of the bending moments are recorded. For example, in the second row of the matrix L_M the **ordinates of influence line** M_2 are recorded.

In the second column of the matrix L_M, the **ordinates** of the **bending moments diagram** M_2, calculated in the regarded cross-sections of the beam loaded by the dimensionless force $\bar{F}_2 = 1$, are recorded.

Consequently, the influence matrix can be formed in two ways: 1) by columns – using unit force diagrams; 2) by rows – using influence lines for efforts.

When calculating the transverse and longitudinal forces, the equations have the form:

$$\vec{Q} = L_Q \vec{F}; \tag{3.18}$$

$$\vec{N} = L_N \vec{F}. \tag{3.19}$$

In equations (3.18) and (3.19) L_Q and L_N are the influence matrices, respectively, of shear and longitudinal forces.

Note that when forming the influence matrix of shear forces L_Q, the calculating cross-sections must be taken to the left-hand and to the right-hand of each concentrated force.

Generally, a beam or other structure can be loaded not only with concentrated forces, but also with distributed loads or concentrated moments. It is possible to construct a matrix of influence that takes into account these types of loads. However, the computational process in this case will become more complicated, the universal character of the computational algorithm will be lost. Therefore, it is recommended that such loads should be converted by bringing them to equivalent concentrated forces according to the general rules of mechanics. When using the load nodal transfer method for this purpose, the position of the nodes is assigned depending on the features of the given load. The spacing of the nodes may be regular or irregular. With a small step length, the accuracy of the calculation increases, but the dimension of the problem increases. In addition to the nodes in the spans of beams, their location above the hinges and supports should be provided.

E x a m p l e. For the beam shown in Figure 3.20, a, we compose the influence matrix of bending moments, calculate bending moments in the calculated sections, plot the diagrams of bending moments caused by the given load and the equivalent concentrated load, compare them.

The positions of the cross-sections are shown in the beam scheme. With a formal approach to the calculation, the position of the required cross-sections should be assigned not only in the spans of the beam, but also where obviously known that bending moments are equal to zero (in this example, cross-sections 1, 5, 9). The load converted to concentrated forces is shown in Figure 3.20, b. The influence matrix of bending moments will have the order (9×9):

$$L_M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4/3 & 2/3 & 0 & -1/2 & -1/3 & -1/6 & 0 & 2/9 \\ 0 & 2/3 & 4/3 & 0 & -1 & -2/3 & -1/3 & 0 & 4/9 \\ 0 & 0 & 0 & 0 & -3/2 & -1 & -1/2 & 0 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 & 0 & -2/3 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1 & 0 & -4/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The explanations for the matrix formation: the values of the ordinates of the ***diagram*** M_9 *(Figure 3.20, d) are recorded in the ninth* ***column*** *of the matrix, the values of the ordinates of the* ***influence line*** *for* M_2 *(Figure 3.20, d) are recorded in the second* ***row****.*

Performing the load transformation, we get the vector of concentrated forces in the form:

$$\vec{F} = [F_1;\ F_2;\ F_3;\ F_4;\ F_5;\ F_6;\ F_7;\ F_8;\ F_9]^T =$$
$$= [0;\ 45;\ 45;\ 17.5;\ 15;\ 35;\ 35;\ 17.5;\ 10]^T\ kH.$$

Having preliminary information that the bending moments are equal to zero in sections 1, 5, and 9, we can delete the corresponding rows of the matrix L_M. Since the concentrated forces above the supports do not affect the outline of the diagram of moments, columns 1, 4, and 8 can be deleted in the matrix. As a result, we obtain a matrix L_M of size (6 × 6):

$$L_M = \begin{bmatrix} 4/3 & 2/3 & -1/2 & -1/3 & -1/6 & 2/9 \\ 2/3 & 4/3 & -1 & -2/3 & -1/3 & 4/9 \\ 0 & 0 & -3/2 & -1 & -1/2 & 2/3 \\ 0 & 0 & 0 & 1 & 1/2 & -2/3 \\ 0 & 0 & 0 & 1/2 & 1 & -4/3 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

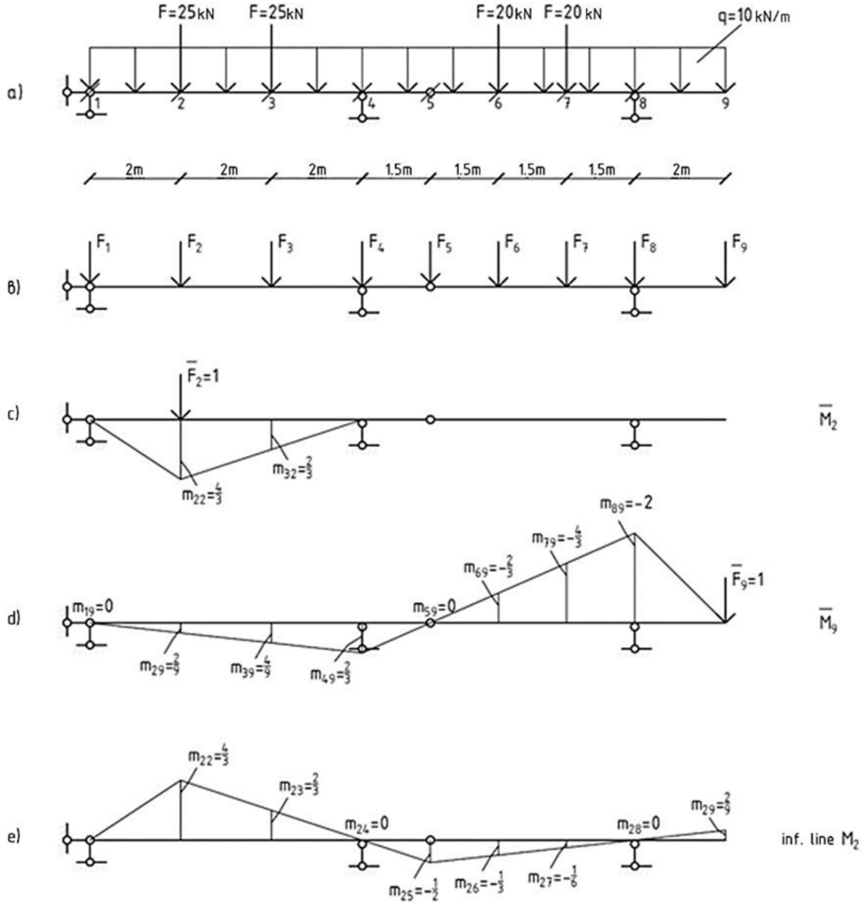


Figure 3.20

The corresponding load vector has the form:

$$\vec{F} = [F_2;\ F_3;\ F_5;\ F_6;\ F_7;\ F_9]^T = [45, 45, 15, 35, 35, 10]^T\ kN.$$

The vector of bending moments in the cross-sections is calculated by the formula (3.16):

$$\vec{M} = [M_2;\ M_3;\ M_4;\ M_6;\ M_7;\ M_8]^T =$$

$$= [67.22;\ 44.44;\ -68.33;\ 45.83;\ 39.17;\ 20.00]^T\ kNm.$$

Figure 3.21, a shows the diagrams of bending moments in the beam with a given load. Figure 3.21, b shows one in the beam with a converted load. The ordinates in the considering cross-sections are the same.

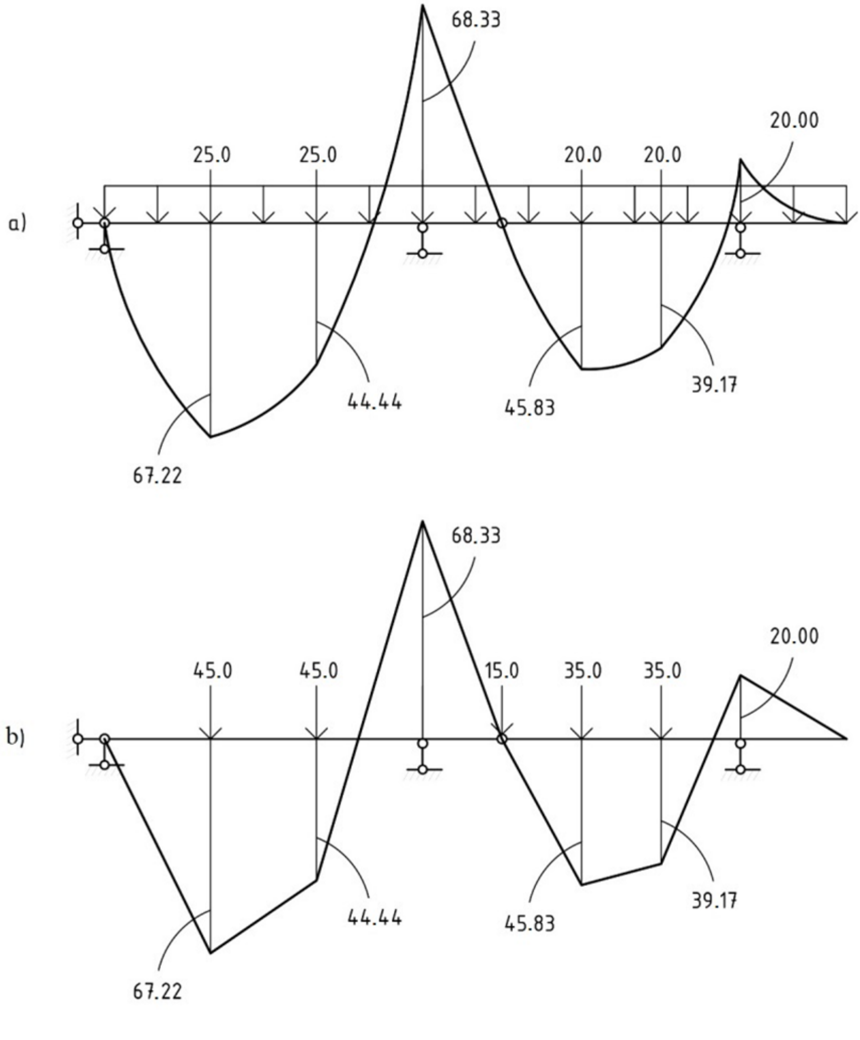


Figure 3.21

THEME 4. CALCULATING OF THREE-HINGED ARCHES AND FRAMES

4.1. General Information and Principles of Creation

A system consisting of two disks interconnected by a hinge and joined with the ground using immovable hinged supports is called a three-hinged system (Figure 4.1).

Three-hinged systems where discs are represented by polygonal bars are called three-hinged frames (Figure 4.2).

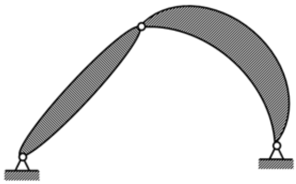


Figure 4.1

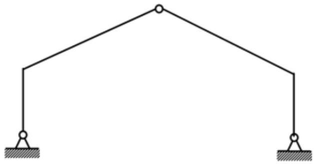


Figure 4.2

Three-hinged systems where the disks are represented by curved bars are called three-hinged arches (Figure 4.3). According to their shape, arches are divided into circular, parabolic, sinusoidal, etc. arches.

Three-hinged systems are formed by the triangles method. Therefore, they are geometrically unchangeable and statically determinate. All three-hinged systems belong to the class of thrusting systems (Figures 1.24, 4.1–4.3).

To eliminate the effect of the horizontal pressure due to the thrust on the underlying structures, the supporting hinges of the three-hinge systems can be connected by horizontal hinged rods or ties. In such cases, one of the supports should be hinged movable. For example, a three-hinged arch with a tie (or a tightrope) at the level of the supports is shown at Figure 4.4.

Three-hinged arches with a tie are externally non-thrusting systems. A vertical loads cause only vertical reactions in supports of such arches.

Arches with an elevated (Figure 4.5) or polygonal complex tie (Figure 4.6) are applied in order to rationally use the space under the arches.

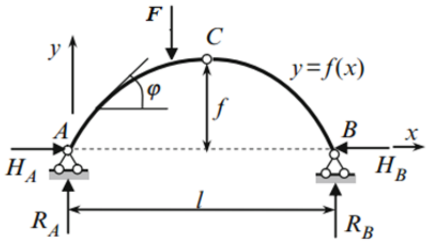


Figure 4.3

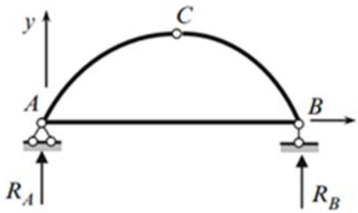


Figure 4.4

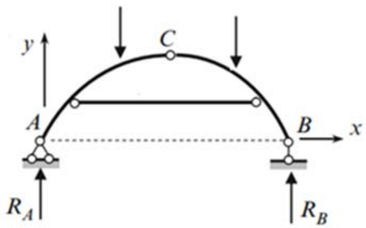


Figure 4.5

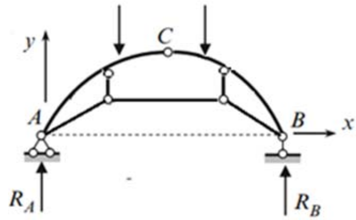


Figure 4.6

4.2. Determining Reactions and Internal Forces in Three-Hinged Arches

Consider a symmetrical three-hinged arch with supports at the same level, loaded with vertical force (Figure 4.7, a).

We compose the equilibrium equation in the form of the sum of the projections of all external forces on the horizontal axis:

$$\sum X = H_A - H_B = 0.$$

From this equilibrium equation it follows that:

$$H_A = H_B = H.$$

That is, the horizontal reactions of the three-hinged arch with the vertical load are opposite in direction, identical in value and equal to the unknown value of H. This value of H and the horizontal reactions themselves are called the three-hinged arch thrust.

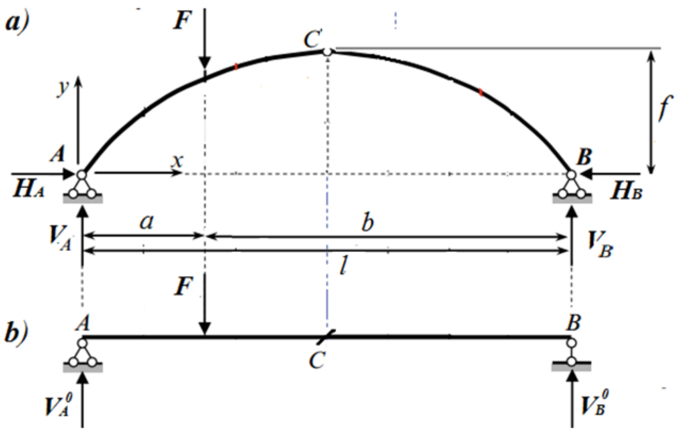


Figure 4.7

Three reactions of the arch: V_A, H_A *and* H_B, intersect at the support point A. Therefore, the vertical reaction V_B of the arch can be determined from the sum of the moments of all external forces relative to this point A.

$$\Sigma M_A = Fa - V_B l = 0, \text{ from } V_B = \frac{Fa}{l} = V_B^0.$$

The resulting expression for determining the vertical reaction V_B of the arch (Figure 4.7, a) is completely equivalent to the expression that can be obtained for determining the vertical reaction of a simple single-span articulated beam (Figure 4.7, b). Such a beam is called equivalent relative to the arch. An equivalent beam has the same span and the same vertical load as the arch.

Accordingly, from the sum of all external forces moments relative to the support point *B*, the vertical reaction V_A of the support A can be found.

$$\Sigma M_B = V_A l - F(l-a) = 0, \text{ from } V_A = \frac{F(l-a)}{l} = V_A^0.$$

Consequently, the vertical reactions of the three-hinged arch under vertical load are equal to the vertical reactions of the equivalent beam.

Therefore, vertical reactions of the arch are often referred to as beam reactions. And this is true with arbitrary vertical load.

Three independent equilibrium equations have already been used to determine the support reactions of the arch. The equilibrium equation in the form of the sum of all external forces projections on the vertical axis is usually used to verify the correctness of the vertical reactions calculation.

$$\Sigma Y = V_A + V_B - F = 0.$$

There is just a need to find the value H of the arch thrust. To determine the arch thrust, we will use the distinguishing property of the arch compared to the equivalent beam. In the intermediate hinge C of the arch (Figure 4.7, a) there is no bending moment. There is no hinge in the corresponding cross-section of the equivalent beam, and the bending moment in this cross-section of the beam (Figure 4.7, b), in the general case, is not equal to zero.

Therefore, defining the bending moment in the hinge C of the arch as the sum of the moments relative to this cross-section of all external forces, for example, located to the left of it, we must equate the resulting expression to zero.

$$M_C = \Sigma M_C^{left} = V_{\mathrm{A}} \frac{l}{2} - F(\frac{l}{2} - a) - Hf = 0.$$

Taking into account, that

$$V_A \frac{l}{2} - F(\frac{l}{2} - a) = M_C^0,$$

where M_C^0 is the bending moment in the cross-section C of the equivalent beam, we can eventually find the thrust H.

$$H = \frac{M_C^0}{f}.$$

Thus, the arch thrust is directly proportional to the beam bending moment in cross-section C of the equivalent beam and inversely proportional to the rise of the arch in the intermediate hinge.

To check the calculated thrust value, the beam bending moment in the cross-section C is usually calculated once again through the sum of the moments of external forces applied to the beam to the right of this section. For our example, it is possible to write

$$M_C^0 = -\Sigma M_C^{right} = V_{\mathrm{B}} \frac{l}{2}.$$

After calculating the support reactions, the determination of the internal forces in the cross-sections of three-hinged arches is usually carried out by the section method, as in any other bars systems.

Consider the features of applying the section method to a three-hinged arch with supports at the same level (Figure 4.7, a). To do this, we cut the arch at some cross-section *x-x* and consider the equilibrium of the left-hand part (Figure 4.8). The action of the discarded right-hand part is replaced by three internal forces: bending moment M_x, transversal force Q_x, and longitudinal (normal) force N_x.

The bending moment M_x in the cross-section *x-x* of the arch is calculated as the sum of the moments of only external forces acting on the left part of the arch relative to the center of gravity of the cross-section *x-x* of the arch

$$M_x = \sum M_x^{left} = V_A x - F\left(x - a_{Fi}\right) - Hy.$$

Taking in to account, that

$$V_A x - F(x-a) = M_x^0,$$

where M_x^0 is the bending moment in the cross-section *x-x* of the equivalent beam (Figure 4.7, b), the bending moment in the cross-section *x-x* of the arch may be finally found using a formula:

$$M_x = M_x^0 - Hy.$$

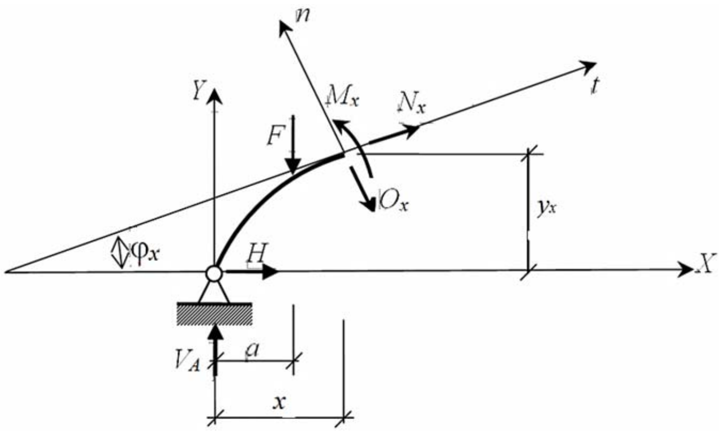


Figure 4.8

The obtained expression shows that the bending moments in the arch are less than the bending moments in the equivalent beam.

It is possible to say that bending moments in the arch have been obtained by algebraic summation of the bending moments in the equivalent beam and the bending moments in the arch, caused by the action of the thrust H only that is seen as two mutually balanced forces applied to the curvilinear bar. The diagram of bending moments due to only the thrust repeats the outline of the arch axis, while the thrust itself serves as a proportionality coefficient.

The bending moments in the beam due to a vertically downward directed load are always positive. Bending moments in the arch from a thrust directed inside the span are always negative. Therefore, the thrust creates an unloading effect for the arch.

We find the transversal force in the *x-x* section of the arch from the sum of the projections of all the forces applied to the left part of the arch (Figure 4.8), normal to the axis of the arch in the section under consideration. Solving the resulting equation relative to Q_x, we obtain

$$Q_x = V_A \cos\varphi_x - F\cos\varphi_x - H\sin\varphi_x =$$
$$= (V_A - F)\cos\varphi_x - H\sin\varphi_x,$$

or

$$Q_x = Q_x^0 \cos\varphi_x - H\sin\varphi_x.$$

Thus, the transversal force in the cross-sections of the arch is expressed through the projection of the beam transversal force Q_x^0 in the corresponding cross-section of the equivalent beam and the projection of the thrust H on the normal to the arch axis in the considered cross-section of the arch.

Similarly, from the sum of the projections of all the forces on the axis tangent to the axis of the arch in section x-x, we find the longitudinal force in this section of the arch

$$N_x = -V_A \sin\varphi_x + F\sin\varphi_x - H\cos\varphi_x = \\ = -(V_A - F)\sin\varphi_x - H\cos\varphi_x,$$

or

$$N_x = -Q_x^0 \sin\varphi_x - H\cos\varphi_x.$$

The longitudinal force in the cross-section of the arch is also expressed through the projection of the beam transversal force Q_x^0 in the corresponding cross-section of the equivalent beam and the projection of the thrust H on the tangent to the arch axis in this cross-section of the arch.

Compared with simple beams in three-hinged arches, the transversal forces, as well as bending moments, are much smaller. But unlike the beams, longitudinal compressive forces occur in the cross-sections of the arches. While no longitudinal forces are present in simple horizontal beams with vertical loads.

The final diagrams of the internal forces in the arch along its entire length would be curvilinear. Curvilinear diagrams, like any graphs, can be built by calculating the values of the corresponding internal forces in a number of predetermined (characteristic) cross-sections of the arch (the more sections are presented the more accurate the diagram).

Let us illustrate the definition of reactions and internal forces using the example of a circular three-hinged arch with a span of l = 36 m with a rise of f = 8 m (Figure 4.9). The arch is loaded with a concentrated force F = 24 kN and a uniformly distributed load q = 2 kN/m.

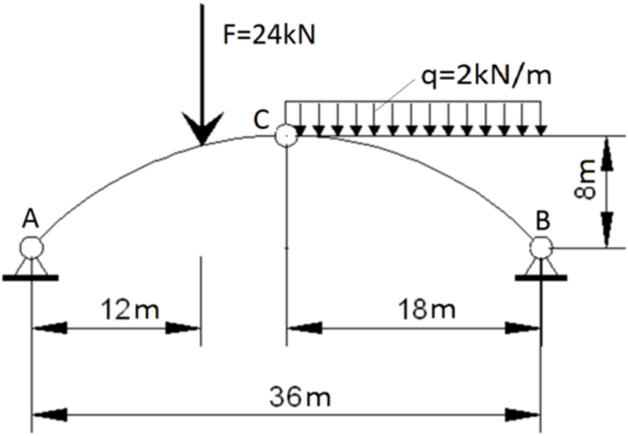


Figure 4.9

The equation of the arch axis, i.e., the equation of the circle arc passing through three points A, C and B, is described by the expression

$$y(x) = f - R + \sqrt{R^2 - (\frac{l}{2} - x)^2}.$$

The radius *R* of the circle and the trigonometric functions of the angle of inclination of the tangent to the axis of the arch are calculated by the formulas:

$$R = \frac{4f^2 + l^2}{8f}, \quad \sin\varphi(x) = \frac{l - 2x}{2R}, \quad \cos(x) = \frac{R - f + y}{R}.$$

The vertical reactions of the arch supports are calculated with the formulas:

$$V_A = V_A^0 = \frac{24 \cdot 24 + 2 \cdot 18 \cdot 9}{36} = 25kN;$$

$$V_B = V_B^0 = \frac{24 \cdot 12 + 2 \cdot 18 \cdot 27}{36} = 35kN.$$

The sum of the projections of all external forces on the vertical axis confirms the result:

$$\sum Y = 25 + 35 - 24 - 2 \cdot 18 = 60 - 60 = 0.$$

In the cross-section *C*, the bending moment of the beam is calculated and checked:

$$M_C^0 = \sum M_C^{left} = 25 \cdot 18 - 24 \cdot 6 = 306\,kN \cdot m,$$
$$M_C^0 = -\sum M_C^{right} = 35 \cdot 18 - 2 \cdot 18 \cdot 9 = 306\,kN \cdot m.$$

Then the arch thrust is calculated:

$$H = \frac{M_C^0}{f} = \frac{306}{8} = 38.25\,kN.$$

To plot the diagrams of the internal forces it is necessary to assign characteristic arch cross-sections. Firstly, these are the supports A and B and the intermediate hinge C. Secondly, these are the point of application of concentrated force and the beginning and the end of the arch segment where the distributed load acts. Thirdly, these are additional intermediate cross-sections necessary for constructing curvilinear segments of the diagrams with sufficient accuracy. In this example, there are at least seven of these characteristic points.

They are located along the arch span in increments of 6 m. To plot the diagrams of the transversal and longitudinal forces at the point of application of the concentrated external force, it is necessary to consider two infinitely close points: one to the left of the application point of the external force, the second to the right of this point. At this cross-section, there will be a jump on the indicated diagrams of the internal forces, and a fracture on the diagram of bending moments. When constructing diagrams of internal forces and moments, it is necessary to monitor their correspondence with each other and the load. The differential depen-dencies between bending moments, transversal forces, and the load must be fulfilled.

To determine the geometric characteristics of the arch, calculate the value of the arch axis radius

$$R = \frac{4 \cdot 8^2 + 36^2}{8 \cdot 8} = 24.25\,m.$$

All further calculations are summarized in the following tables. Calculations in tables can be performed on a calculator, plotting manually, using patterns and other drawing tools. But it is possible to use computers: universal mathematical and engineering software, programming languages, tabular and graphic editors and other modern software tools that automate the process of computing and plotting graphic objects.

Table 4.1

Calculation of bending moments in a three-hinged arch

№ sec	x	Y	M_x^0	$-Hy$	M_x
A	0	0	0	0	0
1	6	4.823	150	-184.47	-34.47
2	12	7.246	300	-277.16	22.84
C	18	8.000	306	-306.00	0
3	24	7.426	276	-277.16	-1.16
4	30	4.823	174	-184.47	-10.47
B	36		0	0	0

So, to build the below diagrams of internal forces in the arch, modern software was used that automates the process of performing calculations and graphing. The diagram of bending moments (Figure 4.10), the diagram of transversal forces (Figure 4.11) and the diagram of longitudinal forces (Figure 4.12), are built on the horizontal projection of the arch axis using the graphic software. Of course, the number of characteristic cross-sections along the span has to be significantly increased.

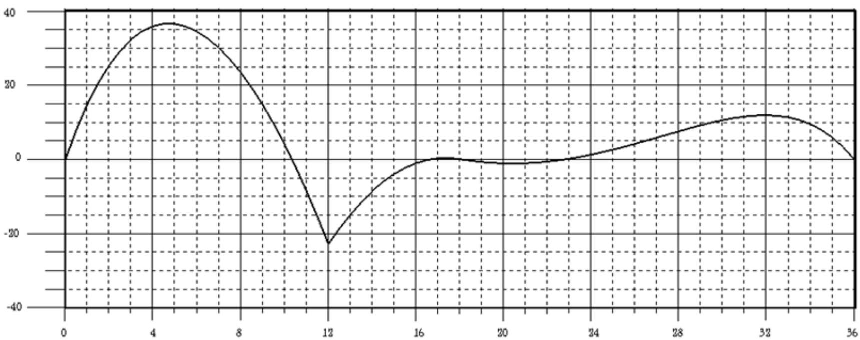


Figure 4.10. Diagram M

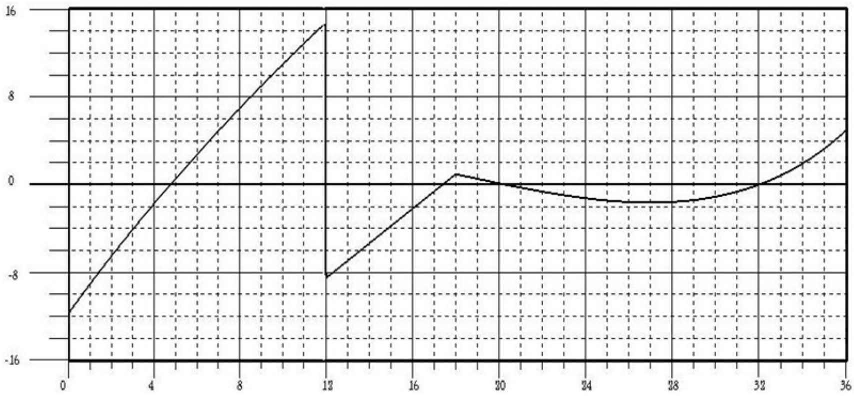


Figure 4.11. Diagram Q

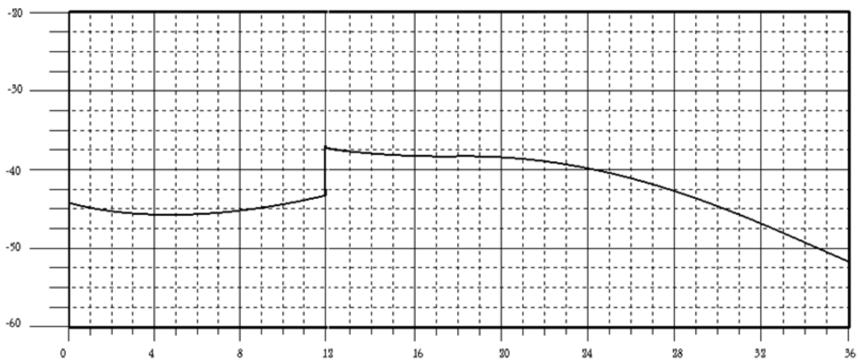


Figure 4.12. Diagram N

As shown in Table 4.1, bending moments in a three-hinged circular arch at a given load are an order of magnitude smaller than bending moments in an equivalent beam. In the support joints and in the intermediate joint, the bending moments in the arch are equal to zero. At the point of application of concentrated force on the diagram of bending moments in the arch, a “beak”-type fracture is observed. On the diagrams of the transversal and longitudinal forces, there are jump discontinuities of the first type: $F\cos\varphi_2$ on the transversal forces diagram Q and $F\sin\varphi_2$ on the longitudinal forces diagram N.

At the points where the transversal forces diagram passes through zero, there are the extremums on the diagram of bending moments. At the point where the segment of the distributed load begins, there is a fracture on the diagram of the transversal forces. In areas where the transversal forces diagram is ascending, the bending moments diagram is convex up. In areas where the transversal forces diagram is downward, the bending moments diagram is convex down.

Such conclusions follow from the differential dependences known from the resistance of materials, according to which the transversal force in the cross-sections of the arch is the first derivative along the length of the arch arc from the function of bending moments. And the load is the first derivative from the transversal force function.

Table 4.2

Arch parameters for calculation of the
transversal and longitudinal forces in the arch

№ sec	x	$\sin\varphi_x$	$\cos\varphi_x$	Q_x^0
A	0	0.7423	0.6701	25
1	6	0.4948	0.8690	25
2_{left}	12	0.2474	0.9689	25
2_{right}	12	0.2474	0.9689	1
C	18	0	1	1
4	24	-0.2474	0.9689	-11
5	30	-0.4948	0.8690	-23
B	36	-0.7423	0.6701	-35

Table 4.3

Calculation of the transversal and longitudinal forces

№ sec	$Q_x^0 \cos\varphi_x$	$-H\sin\varphi_x$	Q_x	$-Q_x^0 \sin\varphi_x$	$-H\cos\varphi_x$	N_x
A	16.753	-28.39	-11.639	-18,557	-25.63	-44.19
1	21.72	-18.928	2.797	-12.371	-33.24	-45.61
2_{left}	24.22	-9.464	14.756	-6.185	-37.06	-43.25
2_{right}	0.9689	-9.464	-8.495	-0.2474	-37.06	-37.31
C	1.0000	0.000	1.0000	0.0000	-38.25	-38.25
4	-10.658	9.464	-1.194	-2.722	-37.06	-39.78
5	-19.987	18.928	-1.059	-11.381	-33.24	-44.62
B	-23.45	28.39	4.94	-25.98	-25.63	-51.61

4.3. Calculating a Three-Hinged Arch with a Tie

Three-hinged tied arches are externally non-thrusting systems. Vertical loads cause only vertical reactions in supports of such arches. These vertical reactions are determined as in simple beams. The horizontal reaction of their immovable hinged support is equal to zero under vertical loads.

But internally such arches are thrusting systems. Their thrust is an internal longitudinal force in ties.

To determine the tightening force, it is necessary to cut an arch by a section through the key hinge of this arch. For example, in a three-hinged arch with a complex tie it is a cross-cut *1-1* passing through the intermediate hinge *C* (Figure 4.13). The equilibrium equation of the left part of the arch in the form of the sum of the moments of all forces relative to the key hinge C gives a possibility to determine the arch thrust *H*.

$$\sum M_C^{left} = 0; \qquad R_A \frac{l}{2} - H\left(f - f_0\right) = 0.$$

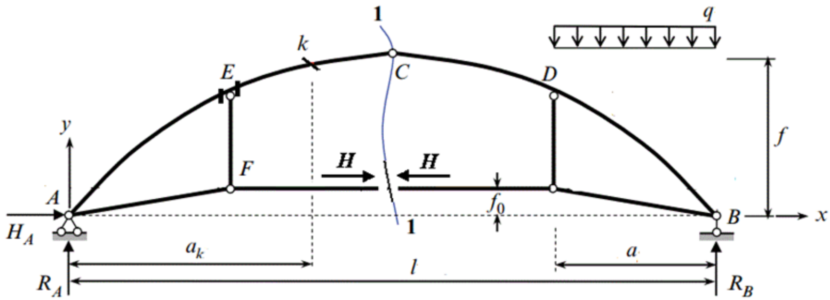


Figure 4.13

The first term in the resulting equation is the bending moment in section C of the equivalent beam:

$$R_A \frac{l}{2} = M_C^0.$$

Therefore, to determine the tightening force (thrust), the following expression can be obtained:

$$H = \frac{M_C^0}{f - f_0}.$$

The internal forces in the other members of the complex tie and in the cross-sections of the arch may be calculated by the usual method of sections.

4.4. Influence Lines in Three-Hinged Arches

Consider an arch loaded with a single vertical force, the position of it is determined by the abscissa x_F (Figure 4.14, a). To determine the vertical support reactions, we compose the equilibrium equations in the form of sums of the moments of all the forces acting on the arch relative to the left and right supports:

$$\Sigma M_A = 0; \qquad 1x_F - R_B l = 0;$$

$$\Sigma M_B = 0; \qquad -1(l - x_F) + R_A l = 0.$$

From these equations we find the functions of changing the vertical support reactions depending on the position of the unit force

$$R_B = \frac{x_F}{l}; \qquad R_A = \frac{l - x_F}{l}.$$

The obtained dependences of the change in the values of the support reactions completely coincide with the corresponding dependences for the support reactions of a simple two-support beam. Therefore, the influence lines for the vertical reactions (Figure 4.14, c, d) in the arch coincide with the influence lines (Inf. Lin.) for the reactions in the corresponding equivalent beam (Figure 4.14, b).

The thrust H of the arch under the action of vertical loads is determined by the expression:

$$H = \frac{M_C^0}{f}.$$

Hence

$$Inf.\,Lin.\ H = \left(Inf.\,Lin.\ M_C^0\right)\Big/ f.$$

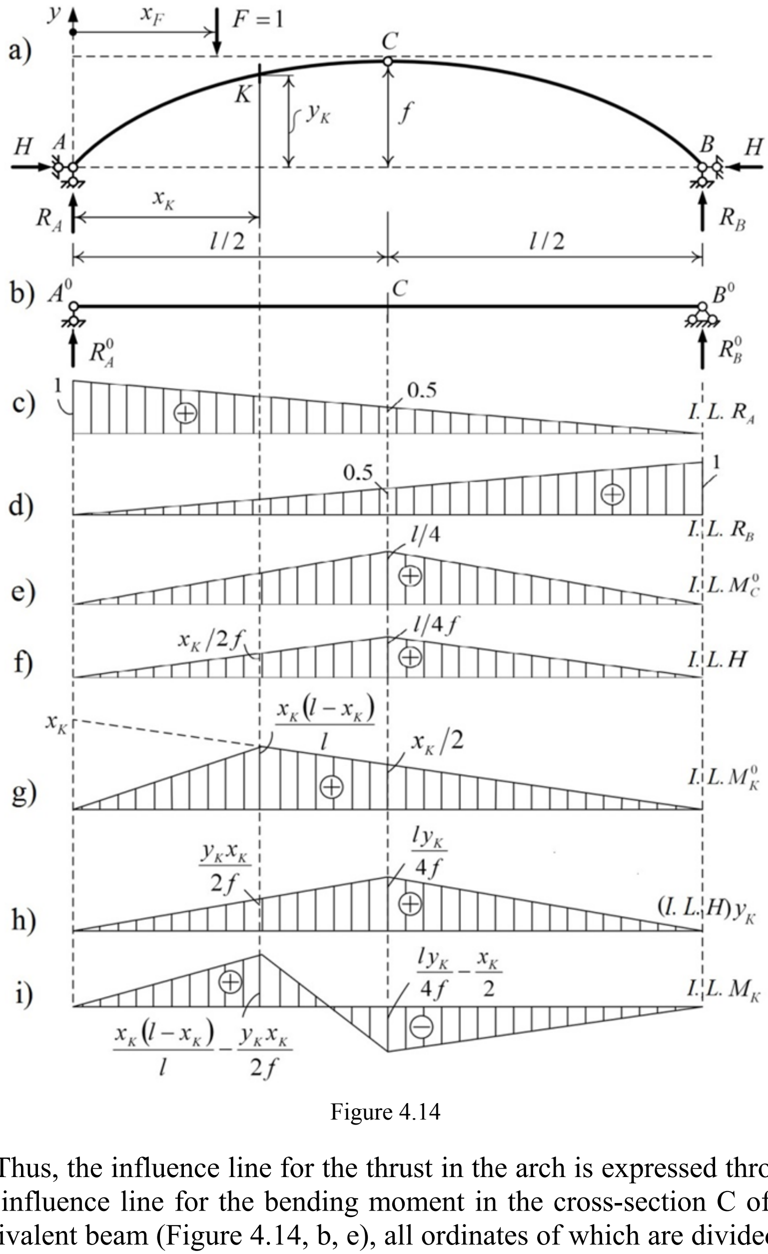


Figure 4.14

Thus, the influence line for the thrust in the arch is expressed through the influence line for the bending moment in the cross-section C of the equivalent beam (Figure 4.14, b, e), all ordinates of which are divided by the value of the arch rise f (Figure 4.14, f).

The influence lines of internal forces in the cross-sections of the arches will be built using the previously obtained dependencies expressing the internal forces in the arches through the corresponding internal beam forces and the arch thrust.

So the bending moment in the section K of the arch (Figure 4.14, a) is determined by the expression

$$M_K = M_K^0 - Hy_K.$$

Since the ordinate y_K of the cross-section *K* of the arch is constant, for the influence line for M_K we get

$$Inf.\,Lin.\,M_K = \left(Inf.\,Lin.\,M_K^0\right) - \left(Inf.\,Lin.\,H\right) y_K.$$

In accordance with this expression, we separately construct the influence line for the bending moment $Inf.Lin.M_K^0$ in the section *K* of the equivalent beam (Figure 4.14, g) and the influence line for the thrust H $Inf.Lin.H$, multiplied by a factor y_K (Figure 4.14, h). Subtracting the ordinates of the second influence line from the ordinates of the first, we get the influence line for the bending moment in the section K of the arch $Inf.Lin.M_K$ (Figure 4.14, i).

The transversal force in the cross-section *K* of the arch is determined by the dependence

$$Q_K = Q_K^0 cos\varphi_K - H\sin\varphi_K.$$

Therefore, the influence line for the transversal force in this arch section can be represented as follows:

$$Inf.Lin.\,Q_K = \left(Inf.Lin.\,Q_K^0\right)\cos\varphi_K - \left(Inf.Lin.H\right)\sin\varphi_K.$$

We build the influence line for the thrust *H* (Figure 4.15, b), and the influence line for the transversal force in the section *K* of the equivalent beam (Figure 4.15, c). Then we build intermediate influence lines, multiplying all the ordinates of $Inf.Lin.Q_K^0$ on $\cos\varphi_K$ (Figure 4.15, d), and the ordinates $Inf.Lin.H$ on $\sin\varphi_K$ (Figure 4.15, e). Subtracting the ordinates of the second from the ordinates of the first influence line, we get the desired influence line for transversal force in the section K of the arch (Figure 4.15, f).

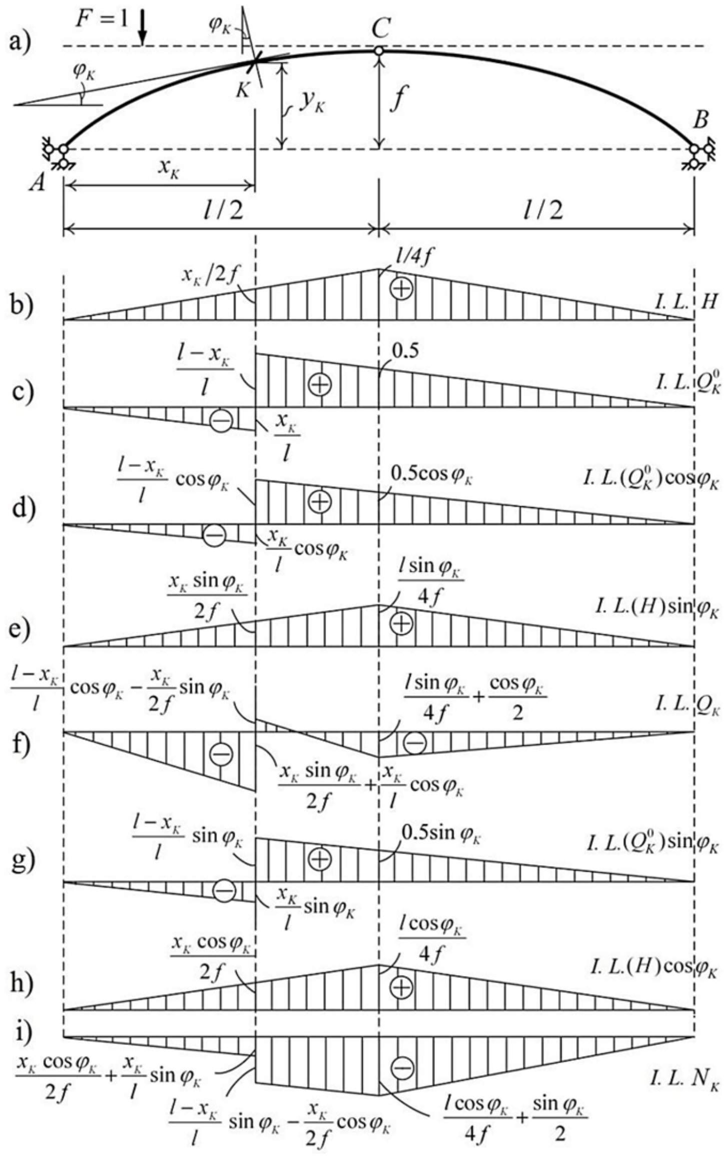


Figure 4.15

The longitudinal force in the cross-section K of the arch is determined by the dependence

$$N_K = -Q_K^0 \sin\varphi_K - H\cos\varphi_K .$$

Accordingly, the expression

$$Inf.Lin.N_K = -\left(Inf.Lin.\,Q_K^0\right)\sin\varphi_K - \left(Inf.Lin.\,H\right)\cos\varphi_K ,$$

is used to construct the influence line for this longitudinal force.

By building intermediate lines of influence $(Inf.Lin.\,Q_K^0)\sin\varphi_K$ (Figure 4.15, g) and $\left(Inf.Lin.\,H\right)\cos\varphi_K$ (Figure 4.15, h), we sum up them. Changing the sign of the result to the opposite, we obtain the desired influence line for the longitudinal force in the section K of the arch (Figure 4.15, i).

4.5. The Rational Axis of the Arch

Rational is called the axis of the arch, if bending moments in the cross-sections of the arch are zeros or close to zeros.

The condition

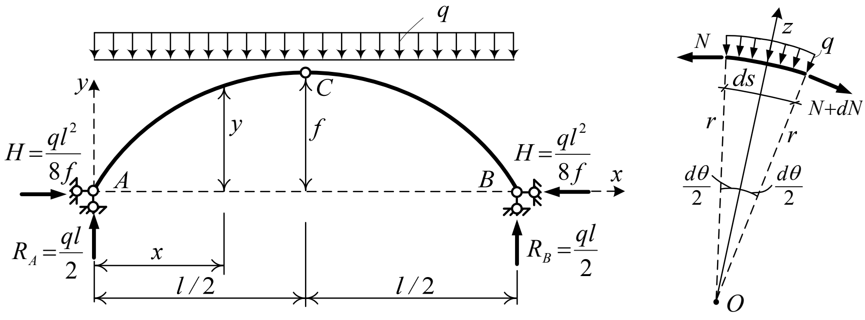
$$M_x = M_x^0 - Hy(x) = 0$$

means that bending moments are absent in all cross sections of the arch. This condition allows you to find the equation of the rational arch axis:

$$y(x) = \frac{M_x^0}{H}.$$

Whence it follows that under the action of vertical loads the ordinates of the rational arch axis are proportional to the bending moments in the equivalent beam having the same span and the same load as the arch. The reciprocal of the thrust H is in this case a proportionality coefficient.

For an example, we define the rational axis of a three-hinged arch when a vertical, evenly distributed load acts on the arch (Figure 4.16, a).



a) Uniformly distributed vertical span load

b) Uniformly distributed radial load

Figure 4.16

The reactions in the arch in this case are equal

$$R_A = R_B = R = \frac{ql}{2}; \qquad H = \frac{ql^2}{8f}.$$

The bending moment in an arbitrary cross-section x of the equivalent beam is defined as the sum of the moments of external forces applied to the beam to the left of section x:

$$M_x^0 = \sum M_x^{left} = \frac{ql}{2}x - (qx)\frac{x}{2} = \frac{qx}{2}(l-x).$$

Dividing the resulting expression by the thrust, we obtain the equation of the rational axis of the three-hinged arch with a uniform load over the span:

$$y(x) = \frac{M_x^0}{H} = \frac{qx(l-x)}{2}\frac{8f}{ql^2}.$$

Or finally

$$y(x) = \frac{4f}{l^2}(lx - x^2).$$

The resulting equation is a quadratic parabola equation. A parabolic arch with a load evenly distributed over the span does not have bending moments. Only longitudinal forces occur in the arch cross-sections.

In the key (in the middle of the span) of the arch, the longitudinal force is

$$N_C = -H = -\frac{ql^2}{8f}.$$

In heels (supports), the longitudinal forces are equal

$$N_A = N_B = \sqrt{R^2 + H^2} = \frac{ql}{2}\sqrt{1 + \frac{l^2}{16f}}.$$

If the arch is outlined in a circle arc, then from the equilibrium conditions of an infinitesimal arch element of length *ds*, it can be proved that the arch circular axis will be rational when the arch is loaded with a uniformly distributed radial load (Figure 4.16, b). With a uniform radial load in a circular arch, there are no bending moments, and the longitudinal forces will be constant along the length of the arch and equal

$$N = -qr.$$

We invite the reader to carry out the corresponding evidence independently.

4.6. Three-Hinged Arches with a Superstructure

Arches that serve as supporting structures for bridges usually have over-the-top or under-arch superstructures. The moving load on the main structure of such arches is not transmitted directly, but through the auxiliary vertical members (links) at certain points – at nodes.

Three-hinged arches with a superstructure are generally regarded as statically determinate, and are complex systems in which an auxiliary part (over- or under-arch superstructure) rests on the main part (three-hinged arch).

The analysis of systems for a moving load is carried out in the same way as for beams with nodal transfer of the load. Initially, the movement of a unit force F = 1 directly along the axis of the main arch is considered, and the influence lines for the factors under study are constructed. Then ordinates are fixed on these influence lines under the nodes. It can be proved that during load nodal transfer, sections of the influence lines between nodal points will be rectilinear. Therefore, if the ordinates fixed under the nodes are connected by straight lines, then the influence lines adjusted in this way will correspond to the influence lines for arches with a superstructure.

E x a m p l e. For a three-hinged arch with a under-arch superstructure (Figure 4.17, a), draw an influence line of the bending moment in section K.

Solution:

1. First, we construct the influence line for the bending moment M_K^* (Inf. Line M_K^*) as if the unit force F = 1 moved directly along the axis of the three-hinged arch (Figure 4.17, b). To construct this influence line, we will use the arguments presented in Section 4.4:

$$Inf.Line\ M_K^* = \left(Inf.Line\ M_K^0\right) - \left(Inf.Line\ H\right) \cdot y_K.$$

The influence line for the bending moment M_K^0 arising in section K of the equivalence beam is shown in Figure 4.17, c, and the influence line for the thrust H multiplied by y_K is shown in Figure 4.17, d.

For the initial data of the example: with $x_K = 3m$, it follows that:

$$y_K = \frac{4f}{l^2} x_K \left(l - x_K\right) = \frac{4 \cdot 3}{12^2} 3 \cdot \left(12 - 3\right) = 2.25m.$$

The influence line M_K^* obtained by subtracting $\left(inf.line\ H\right) \cdot y_K$ from $inf.line\ M_K^0$ is shown in Figure 4.17, e.

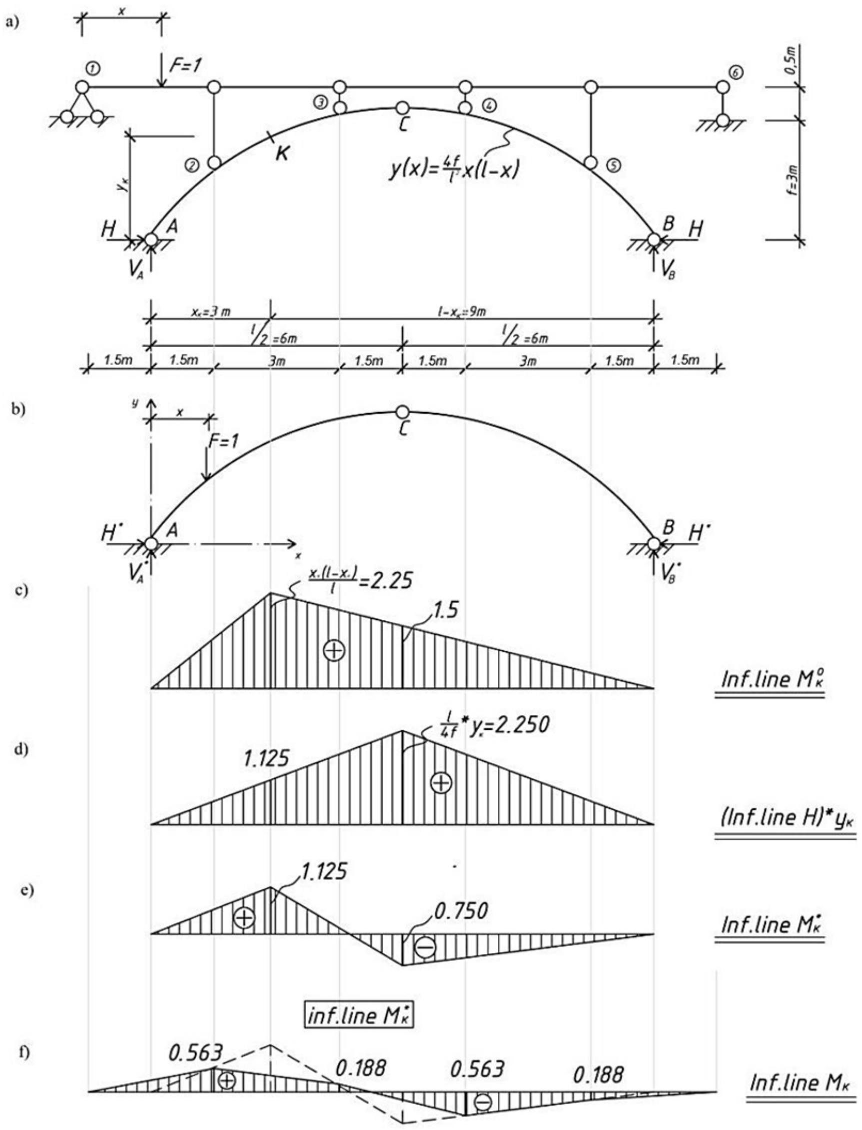


Figure. 4.17

2. We correct the constructed influence line M_K^* taking into account the nodal transfer of the load. To do this, we calculate the ordinates of

this influence line under the nodal points 2, 3, 4 and 5. The ordinates under the nodal points 1 and 6 have zero values (Figure 4.17, f).

3. We connect the calculated ordinates with straight lines. The resulting graph is the influence line for the bending moment in the section *K* (Figure 4.17, f) under the condition that the load on the arch is transmitted through the over-arch superstructure.

4.7. Determining Support Reactions and Internal Forces in Three-Hinged Frames

Consider the process of determining support reactions in a three-hinged frame with supports at different levels (Figure 4.18).

The frame is loaded with a horizontal uniformly distributed load of intensity *q = 4 kN/m* and a vertical concentrated force *F = 12 kN*. The expected directions of support reactions are shown in Figure 4.18.

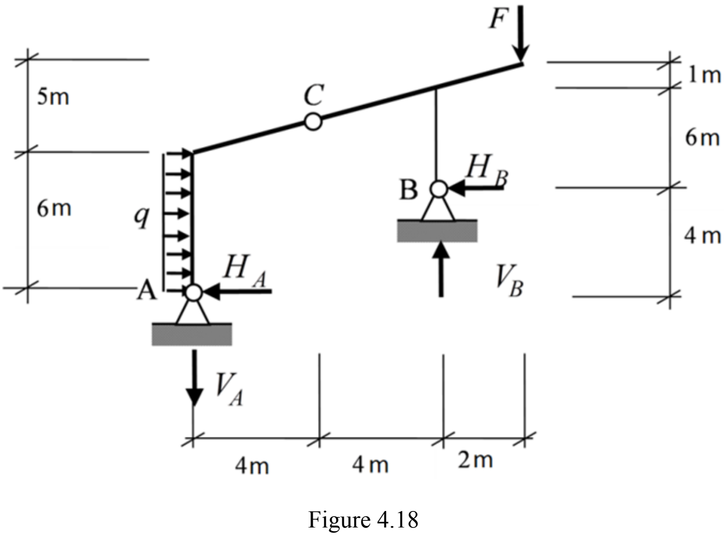


Figure 4.18

As usual, we compose the sum of the moments of all external forces relative to the support *B*:

$$\Sigma M_B = -4 \cdot 6 \cdot 1 + 12 \cdot 2 - V_A \cdot 8 + H \cdot 4 = 0.$$

Since the equation contains two unknown quantities V_A and H_A, we compose the second equation in the form of the sum of the moments of the left forces only, relative to the joint C:

$$\sum M_C^{left} = -4 \cdot 6 \cdot 5 - V_A \cdot 4 + H_A \cdot 8 = 0.$$

The resulting equation includes the same two unknown quantities V_A and H_A. Solving the system of two joint equations, we find the values of the support reactions of the right support A:

$$V_A = 10\,kN; \quad H_A = 20\,kN.$$

Accordingly, the reactions of support B will be found from the sum of the moments of all external forces relative to the support A and the sum of the moments of only the right forces relative to the intermediate joint C:

$$\sum M_A = 4 \cdot 6 \cdot 3 + 12 \cdot 10 - V_B 8 - H_B 4 = 0,$$
$$\sum M_C^{right} = 12 \cdot 6 - V_B 4 + H_B 4 = 0.$$

Solving the resulting system of two equations, we find

$$V_B = 22\,kN; \quad H_B = 4\,kN.$$

The calculated values of all supporting reactions are positive. Therefore, their directions shown in Figure 4.18 are valid.

We will check the results. We compose the sum of all forces projections on the X and Y axes, as well as the sum of all external forces moments relative to, let's say, the point D in the middle of the left rack (the moment from the distributed load and the moment from the reaction V_A at this point are zero, which reduces the amount of calculations):

$$\Sigma X = 4 \cdot 5 - 20 - 4 = 24 - 24 = 0,$$
$$\Sigma Y = -10 + 22 - 12 = -22 + 22 = 0,$$
$$\Sigma M_D = 20 \cdot 3 + 12 \cdot 10 - 22 \cdot 8 - 4 \cdot 1 = 180 - 180 - 0.$$

All three checking equilibrium equations are satisfied identically.

Summing up, we can recommend the following rules for calculating support reactions in arbitrary three-hinged arches and other three-hinged systems with arbitrary external loads.

Usually, four equations are composed to calculate the four support reactions, and three more equations are used to verify the results.

The support reactions of the left support (V_A, and H_A) are calculated from two equations.

The first is the sum of the moments of all external forces relative to the right support B:

$$\Sigma M_B = 0.$$

The second is the sum of the moments relative to the intermediate joint C of only external forces located to the left of the joint C:

$$\Sigma M_C^{left} = 0.$$

The support reactions of the right support (V_B and H_B) are calculated from two more equations.

The third is the sum of all external forces moments relative to the right support A:

$$\Sigma M_A = 0.$$

Fourth is the sum of the moments relative to the intermediate joint C of only external forces located to the right of the joint C:

$$\Sigma M_C^{right} = 0.$$

To verify the results, the sums of all external forces projections on the coordinate axes and the sum of all external forces moments relative to any point not previously used as a moment point are written.

After determining the support reactions, the diagrams of internal forces in the bars of the three-hinged frame are constructed, as in any other bars systems. The calculated support reactions are considered as known external forces. Internal forces are calculated according to general rules in given characteristic cross-sections. For the considered frame from cross-sections with nonzero bending moments, six characteristic sections have been selected (Figure 4.19): this is the beginning and end of each bar, the middle of the distributed load application segment.

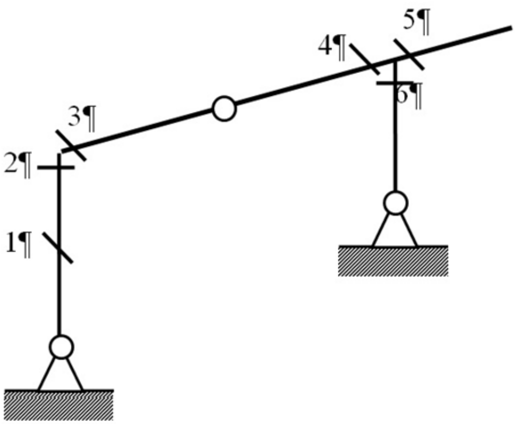


Figure 4.19

We calculate the bending moments in the indicated sections:

$$M_1 = \sum M_1^{\substack{left\\bottom}} = 20\cdot 3 - 4\cdot 3\cdot 1.5 = 42\ kN\cdot m.$$

$$M_2 = \sum M_2^{\substack{left\\bottom}} = 20\cdot 6 - 4\cdot 6\cdot 3 = 48\ kN\cdot m\ .$$

$$M_3 = \sum M_3^{left} = 20\cdot 6 - 4\cdot 6\cdot 3 = 48\ kN\cdot m\ .$$

$$M_4 = \sum M_4^{left} = 20\cdot 10 - 4\cdot 6\cdot 7 - 10\cdot 8 = -48\ kN\cdot m\ .$$

$$M_5 = -\sum M_5^{right} = -(12\cdot 2) = -24\ kN\cdot m\ .$$

$$M_6 = \sum M_6^{\substack{bottom\\left}} = 4\cdot 6 = 24\ kN\cdot m\ .$$

The diagram of bending moments is plotted in Figure 4.20.

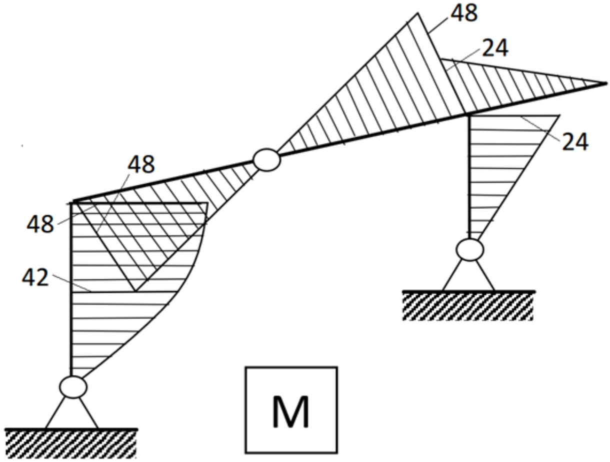


Figure 4.20

We begin the calculation of transversal and longitudinal forces from the support section A:

$$Q_A = H_A = 20\,kN, \quad N_A = V_A = 10\,kN.$$

On a length A-2 of the bar, where a uniformly distributed load is applied, the transversal forces linearly decrease, and the longitudinal forces are constant. Therefore, we calculate:

$$Q_2 = 20 - 4 \cdot 6 = -4\,kN, \quad N_2 = N_A = 10\,kN.$$

On an inclined bar section 3-4, the transversal and longitudinal forces are constant. The tangent of the angle α of bar inclination to the horizon on this length is equal $\mathrm{tg}\alpha = 4/8 = 0.5$. Therefore $\sin\alpha = 0.4472$, and $\cos\alpha = 0.8944$. Next we calculate:

$$Q_3 = Q_4 = -10 \cdot 0.8944 + (20 - 4 \cdot 6) \cdot 0.4472 = -10.73\,kN,$$

$$N_3 = N_4 = -10 \cdot 0.4472 + (20 - 4 \cdot 6) \cdot 0.8944 = 0.8944\,kN.$$

In cross-sections of the inclined console, the transversal and longitudinal forces are also constant. It is enough to calculate them in the cross-section 5 through the right hand external forces:

$$Q_5 = 12 \cdot 0.8944 = 10.73\, kN, \quad N_5 = -12 \cdot 0.4472 = -5.366\, kN.$$

On the right strut, transversal and longitudinal forces are also constant. We calculate:

$$Q_6 = Q_B = 4\, kN, \quad N_6 = N_B = -22\, kN.$$

Diagrams of transversal and longitudinal forces are plotted in Figure 4.21 and Figure 4.22.

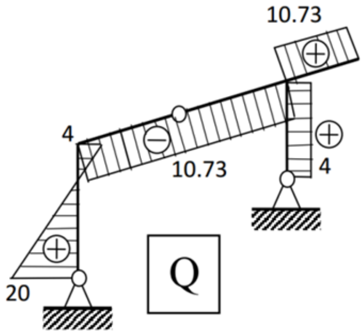


Figure 4.21

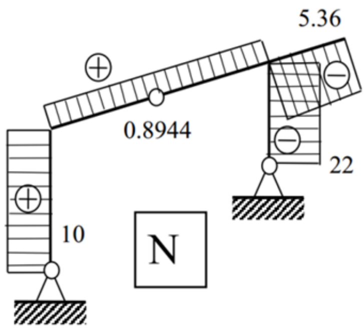


Figure 4.22

Check the equilibrium of the left and right frame nodes (Figure 4.23).

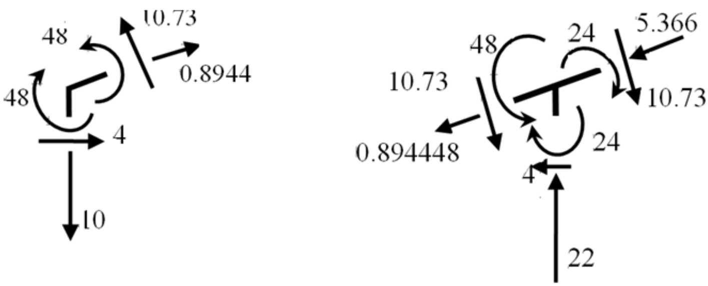


Figure 4.23

For the left node we have:

$$\sum M = 48 - 48 = 0.$$

$$\sum X = 4 - 10.73 \cdot 0.4472 + 0.8944 \cdot 0.8944 = 4.800 - 4.798 = 0.002\ kN.$$

Relative error $\quad \varepsilon = \dfrac{0.002 \cdot 100\ \%}{4.800} = 0.0417\ \% < 3\ \%.$

$$\sum Y = -10 + 10.73 \cdot 0.8944 + 0.8944 \cdot 0.4472 = 9.997 - 10 = -0.003\ kN.$$

Relative error $\quad \varepsilon = \dfrac{|-0.003| \cdot 100\ \%}{10} = 0.03\ \% < 3\ \%.$

For the right node we have:

$$\sum M = -48 + 24 + 24 = 0.$$

$$\sum X = (10.73 + 10.73) \cdot 0.4472 - (0.8944 + 5.367) \cdot 0.8944 - 4 =$$

$$= 9.597 - 9.600 = -0.003\ kN.$$

Relative error $\quad \varepsilon = \dfrac{|-0.003| \cdot 100\ \%}{9.600} = 0.0312\ \% < 3\ \%.$

$$\sum Y = (-10.73 - 10.73) \cdot 0.8944 - (0.8944 + 5.367) \cdot 0.4472 + 22 =$$

$$= -21.993 + 22 = -0.007\ kN.$$

Relative error $\quad \varepsilon = \dfrac{|-0.007| \cdot 100\ \%}{22} = 0.0318\ \% < 3\ \%.$

THEME 5. CALCULATING PLANE STATICALLY DETERMINATE TRUSSES

5.1. Trusses: Concept, Classification

Geometrically unchangeable bars systems composed, as a rule, of rectilinear rods connected at their ends by ideal hinges without friction, are usually called trusses.

Therefore the design scheme of a truss is a geometrically unchangeable system of articulated rods (Figures 5.1–5.3, 5.5).

A real structures with rigid nodes (welded or monolithic), which remain geometrically unchangeable after the mental replacement of all rigid nodes with hinged ones are often also called trusses (Figure 5.4).

The rods located along the upper and lower contours of the truss form its top and bottom chords. The rods connecting the both chords form a lattice of the truss. Inclined rods of the lattice are called diagonals. The vertical rods of the lattice are called struts (or pendants if they are tensile).

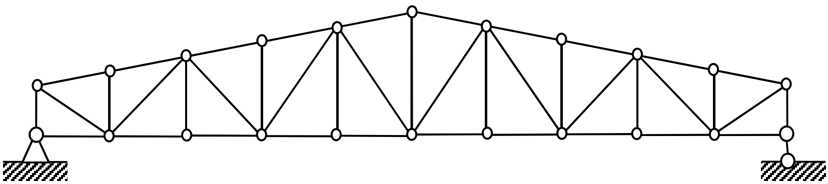


Figure 5.1. Trapezoidal truss with triangular lattice and additional struts

The classification of design schemes of trusses as hinge-rod systems can be carried out according to many criteria.

Trusses, like other structures, are divided into plane (Figures 5.1–5.3) and spatial (Figure 5.4).

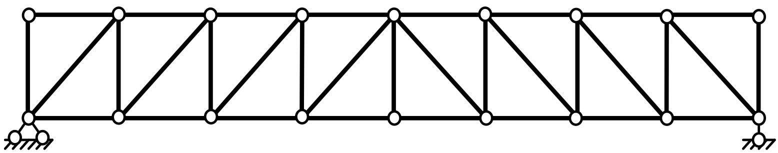


Figure 5.2. Beam truss with parallel chords and a lattice of N form

Trusses can be subdivided according to the supporting conditions into trusses free of thrust, or beam trusses (figures 5.1, 5.2, 5.5); and trusses with thrust, or arch trusses (figure 5.3).

According to the outline of the chords, trusses are divided into trusses with parallel chords (Figure 5.2 and 5.5, a) and polygonal chords (Figure 5.5, b), triangular, trapezoidal trusses (Figure 5.1), parabolic, circular (Figure 5.3), etc.

According to the type of lattice, trusses are divided into trusses with a triangular lattice or of V-form lattice (Figure 5.1; 5.3), trusses with a N-form lattice (Figure 5.2), trusses with a crossed lattice (Figure 5.5, a), trusses with a mixed lattice (Figure 5.5, b).

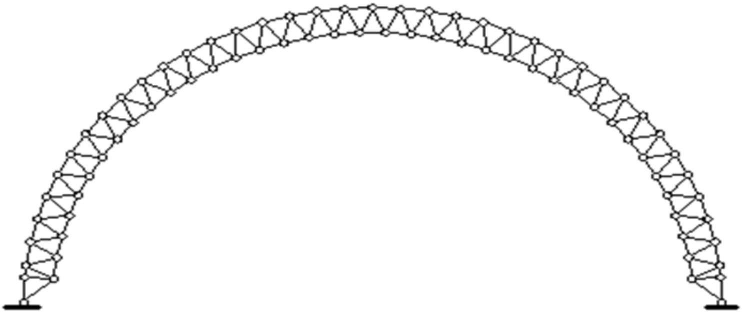


Figure 5.3. Circular two-hinged arch truss with a triangle lattice

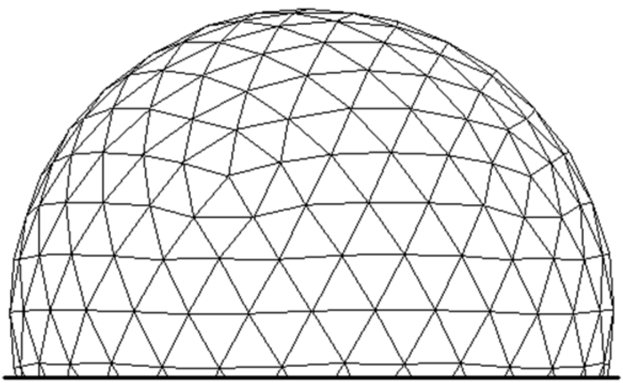
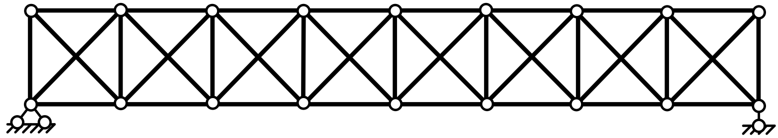


Figure 5.4. Lattice dome

a) beam truss with a crossed lattice



b) simply supported truss with overhang and mixed lattice

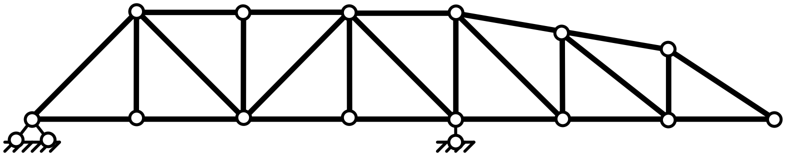


Figure 5.5

The given classification is far from complete. In real buildings, trusses of various types can be used.

5.2. Plane Trusses. Degree of Freedom and Variability

A necessary condition for the geometric immutability and static definability of a truss as a hinge-rod system is that its degree of freedom is equal to zero (W = 0) or, if the truss is separated from its supports, its degree of variability is also equal to zero (V = 0).

We assume that the truss in the general case consists of N nodes interconnected by B truss rods (bars) and attached to the supports with L supporting rods (simple links).

Then, for a plane truss, its degree of freedom W with respect to the reference system associated with the supporting surface is equal to

$$W = 2N - B - L,$$

where $2N$ is the degree of freedom of N free nodes as material points,

B is the number of truss rods (bars) that connects truss nodes as simple links and eliminate B degrees of freedom,

L is the number of simple support rods (links) that also eliminates L degrees of freedom of the system.

The degree of freedom of a plane hinge-rod system, not having support connections and separated from the supports, consists of the degree of freedom of the system as a rigid whole (disk), equal to three (on the plane), and the degree of variability of V of its elements relative to each other (internal mutability). Thus, we can write

$$W = 3 + V,$$

from

$$V = W - 3.$$

Substituting the expression for W under the condition $L = 0$ in the last formula, we obtain the final expression for calculating the degree of variability of the truss (hinged-rod system) disconnected from the supports,

$$V = 2N - B - 3.$$

If the degree of freedom (degree of variability) of the truss is positive (greater than zero)

$$W > 0 \quad (V > 0),$$

then the truss is geometrically variable. The truss structure lacks W links (rods).

If the degree of freedom (degree of variability) of the truss is negative (less than zero)

$$W < 0 \quad (V < 0),$$

then the truss formally contains an excessive number of links (rods) and is, again formally, statically indeterminate.

If the degree of freedom (degree of variability) of the truss is zero

$$W = 0 \quad (V = 0),$$

then the truss formally has the number of rods (links) necessary for geometric immutability and can, again, formally, be statically determinate.

For example, the beam truss (Figure 5.1) has 22 nodes, 10 rods in the top and bottom chords, 10 diagonals and 11 struts. The truss is supported by three support rods. Its degree of freedom:

$$W = 2 \cdot 22 - 41 - 3 = 44 - 44.$$

This means that the truss has the required number of rods and support links for geometric immutability and static definability.

A truss with parallel chords and a cross lattice consists of 18 nodes connected by 41 rods and rests, like a simple beam, on three support rods. Its degree of freedom:

$$W = 2 \cdot 18 - 41 - 3 = 36 - 44 = -8.$$

Therefore, this truss has 8 redundant links and is statically indeterminate.

5.3. Plane Trusses. Formation Methods

As noted in subsection 1.1.5, for a final conclusion on the geometric immutability and on the static definability of a truss, as well as any other bar system, an analysis of its structure and of the laws by which it is compiled are necessary. Trusses of only the correct structure can be really geometrically unchangeable ($W \leq 0$) and statically determinate ($W = 0$).

Trusses (systems) that are partially statically indeterminate and partially geometrically changeable, as well as systems that are instantaneously changeable are relative to systems of irregular structure. For such systems, the concept of the degree of freedom or of variability becomes uncertain, meaningless.

The methods, rules for the formation of trusses of a knowingly correct structure, remain the same as for any other bar systems. Recall the main ones.

1. The degree of freedom of the truss will not change if you attach (disconnect) a node to it using two rods that do not lie on one straight line (dyad method). The rods can be knowingly geometrically unchangeable and statically determinate trusses.

2. Three rods (three disks) connected by three hinges that do not locate on one straight line form an internally geometrically unchangeable system (single disk) without redundant connections.

3. Two trusses (two disks) connected by three rods lying on straight lines, not intersecting all three at once at one point and not parallel each other, form a single system (disk). In such a system, the total number of redundant rods does not change, and the total degree of freedom is reduced by three units.

4. Two trusses (two disks) connected by a common hinge and by a rod that does not pass through a common hinge form a whole truss (disk), while the total number of redundant rods does not increase, and the total degree of freedom decreases by three units.

By their structure, the trusses (Figures 5.1, 5.2 and 5.5, b) composed of rod triangles are disks without redundant connections. These disks are supported by beam supports (in total, three support rods, not parallel, not intersecting at one point). Consequently, all these trusses are geometrically unchangeable and statically determinate.

The arched truss (Figure 5.3) is also composed of rod triangles forming a circular disk. But this disk rests on two immovable hinged supports (in total four support links). Therefore, one of the support links (horizontal) is superfluous. This arch is statically indeterminate.

A truss with a crossed lattice (Figure 5.5, a) differs in its structure from a geometrically unchangeable and statically determinate truss with an *N*-form lattice (Figure 5.2) by the presence of eight additional diagonals. Therefore, additional diagonals represent redundant rods. This truss is geometrically unchangeable, but statically indeterminate eight times.

5.4. Determining Internal Forces in the Truss Rods from Stationary Loads

The determination of internal forces in the rods of plane trusses, as in other systems (beams, frames, arches), is carried out by the method of sections. The essence of the section method for truss is as follows. The truss is cut (divided) into two (Figure 5.6, a) or several parts so that the rod in which the internal force is to be calculated is cut up. For a truss in equilibrium, any part of it must also be in equilibrium. The equilibrium equations compiled for the selected part of the truss, along with external nodal loads, include forces in the rods that are cut up. The internal forces (longitudinal forces) in the rods that are cut up are usually directed from the node to the cut that corresponds to the tension of the rods (Figure 5.6, b). The equilibrium equations must be compiled in such a form and sequence that each of them includes only one unknown force, if it possible. The algebraic signs of the found forces are retained. This allows us to determine the type of stress state of the rod by the sign of effort: tension or compression. The plus sign corresponds to extension in the rod, and the minus sign corresponds to compression in the rod.

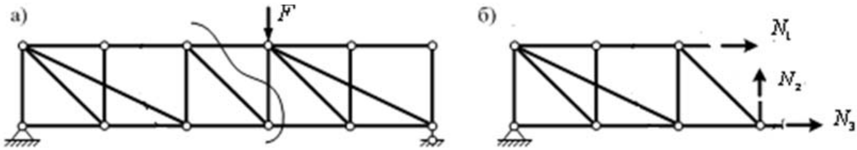


Figure 5.6

Consider the process of applying the section method using the example of a trapezoidal truss with a triangular lattice and additional struts adjacent to the upper chord. The length of the span of the truss is 24 m. The height of the truss above the supports is 2 m. The height of the truss in the middle of the span is 5 m. The truss is loaded with six vertical nodal forces of 24 kN each (Figure 5.7).

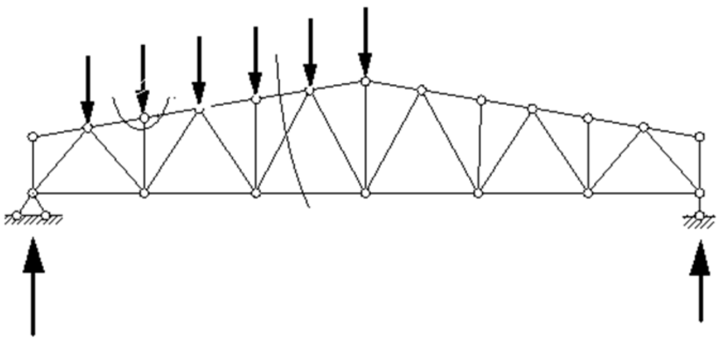


Figure 5.7

First we find reactions. We find the reaction of the left support from the sum of the moments of all forces relative to the right support point:

$$V_A = \frac{24 \cdot (22 + 20 + 18 + 16 + 14 + 12)}{24} = 102\ kN.$$

Accordingly, the reaction of the right support will be found from the sum of the moments of all forces relative to the left support point:

$$V_B = \frac{24 \cdot (2+4+6+8+10+12)}{24} = 42\ kN.$$

Perform a check in the form of the sum of the projections of the active and reactive forces on the vertical axis:

$$\sum Y = 102 + 42 - 6 \cdot 24 = 144 - 144 = 0.$$

The forces in the truss rods can be determined in any order. For example, we make a vertical section through the fifth panel of the upper chord and the third panel of the lower chord, as shown in Figure 5.7. We are considering the equilibrium of the left part.

We find the force N_1 in the cut rod of the third panel of the lower chord from the sum of the moments of all the left forces relative to the moment point where the cut diagonal and the cut rod of the upper chord intersect. The height of the truss at this point at a distance of 10 m from the left support is 4.5 m. This will be the arm of the force N_1 in the cut rod of the lower chord. Solving the equilibrium equation with respect to N_1, we find

$$N_1 = \frac{102 \cdot 10 - 24 \cdot 8 - 24 \cdot 6 - 24 \cdot 4 - 24 \cdot 2}{4.5} = 120\ kN.$$

We find the force N_2 in the cut rod of the upper chord from the sum of the moments of left forces relative to the node (moment point) where the cut diagonal and the cut rod of the lower chord intersect. This node is located at a distance of 8 m from the left support. The height of the truss in this node is 4 m. The arm of the force N_2 in the upper chord relative to this moment point located on the lower chord is equal to the projection of the height of the truss at this point on the normal to the upper chord. We calculate the tangent of the angle θ of inclination of the upper chord to the horizon, and through it the sine and cosine of this angle:

$$\mathrm{tg}\theta = \frac{5-2}{12} = 0.25, \quad \sin\theta = 0.2425, \quad \cos\theta = 0.9701.$$

We calculate the arm ρ_2 of the force N_2 relative to the moment point

$$\rho_2 = 4 \cdot 0.9701 = 3.880.$$

So, from the sum of the moments of the left forces relative to the moment point, we find the force in the cut rod of the upper chord

$$N_2 = -\frac{102 \cdot 8 - 24 \cdot 6 - 24 \cdot 4 - 24 \cdot 2}{3.880} = -136.08\, kN.$$

A negative value of the found force means that the rod of the upper truss chord is compressed.

To find the force N_3 in the cut diagonal, we calculate the sine of the angle α of inclination of this rod to the horizon, and then the cosine of the angle α:

$$\sin\alpha = \frac{4.5}{\sqrt{4.5^2 + 2^2}} = 0.9138, \qquad \cos\alpha = 0.4062.$$

We project on the vertical axis all the forces acting on the left part:

$$\sum Y = 102 - 4 \cdot 24 + (-136.08 \cdot 0.2425) + N_3 \cdot 0.9138 = 0.$$

From where we find

$$N_3 = -\frac{102 - 96 - 33.00}{0.9138} = 29.55\, kN.$$

We will check the calculations by projecting all the forces acting on the left part on the horizontal axis:

$$\sum X = N_1 + N_2 \cos\theta + N_3 \cos\alpha =$$
$$= 120 - 136.08 \cdot 0.9701 + 29.55 \cdot 0.4062 =$$
$$= 120 - 132.01 + 12.00 = -0.01.$$

Verification showed that the calculations were performed almost exactly. The error is only a unit of the fifth significant digit of one of the terms.

In a similar way, internal forces can be found in the remaining rods of the truss. We invite the reader to perform the necessary actions on their own.

In some cases, cuts (sections) may be carried out so that only one node is cut out from the truss. For example, such is the third left node of the upper truss chord (Figure 5.7). The cut out node is shown in Figure 5.8. The performed cut demonstrates a special case of the section method, called the cut-out nodes method. All the forces acting on the one cut-out node converge at one point, in the cut-out node itself. For such a system of forces passing through one point, only two independent equilibrium equations can be compiled. Therefore, the nodes should be cut out in such an order that in each cut out node there were no more than two unknown forces.

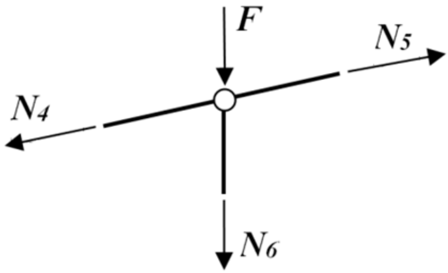


Figure 5.8

For the cut node under consideration from the sum of the projections onto the horizontal axis (Figure 5.8)

$$\Sigma X = -N_4 \cos\theta + N_5 \cos\theta = 0$$

only equality of forces in the rods of adjacent panels of the upper chord follows

$$N_4 = N_5.$$

The values of these forces remain unknown. They will have to be found from other equilibrium equations, for example, as shown above.

But from the sum of the projections onto the vertical axis

$$\Sigma Y = -F - N_6 = 0$$

we easily find

$$N_6 = -f = -24\,kN.$$

Consequently, the second on the left-hand vertical rod of the truss is compressed with a force of 24 kN.

The cut-out nodes method often allows you to visually, without calculation, set the rods with zero forces, rods with the same forces, rods with known forces in advance. Consider these most common special cases of equilibrium of nodes:

1. Double-rod unloaded node (Figure 5.9, a). The forces in both rods are zero.

2. Double-rod node with a load along one of the rods (Figure 5.9, b). The force in the rod lying on one straight line with an external force is equal to this force. The force in the other single rod is zero.

3. Three-rod unloaded node (Figure 5.9, c). The forces in the rods lying on one straight line are equal. The force in the third single rod is zero.

4. A three-rod node with a load along a single rod (Figure 5.9, d). The forces in the rods lying on one straight line are equal. The force in the third single rod is equal to the external force.

5. A four-rod unloaded node with rods lying in pairs on straight lines (Figure 5.9, e). The forces in pairs lying on the same straight line are the same.

6. A four-rod unloaded node with two rods lying on one straight line and with two others, equally inclined to the first two (Figure 5.9, f). The forces in the equally inclined rods are equal in value and opposite in sign.

So, based on the considered particular cases of equilibrium of nodes, it can be immediately established in the considered truss (Figure 5.7) that the vertical rods above the supports and the extreme rods of the upper chord do not loaded (case 1). Two struts of the right-hand half-span of the truss are also not loaded (case 3). The second and third struts of the left-hand half-span are compressed with a force of 24 kN (case 4). The forces in the rods of the upper chord adjacent to the intermediate struts are equal in pairs (case 4 and 3).

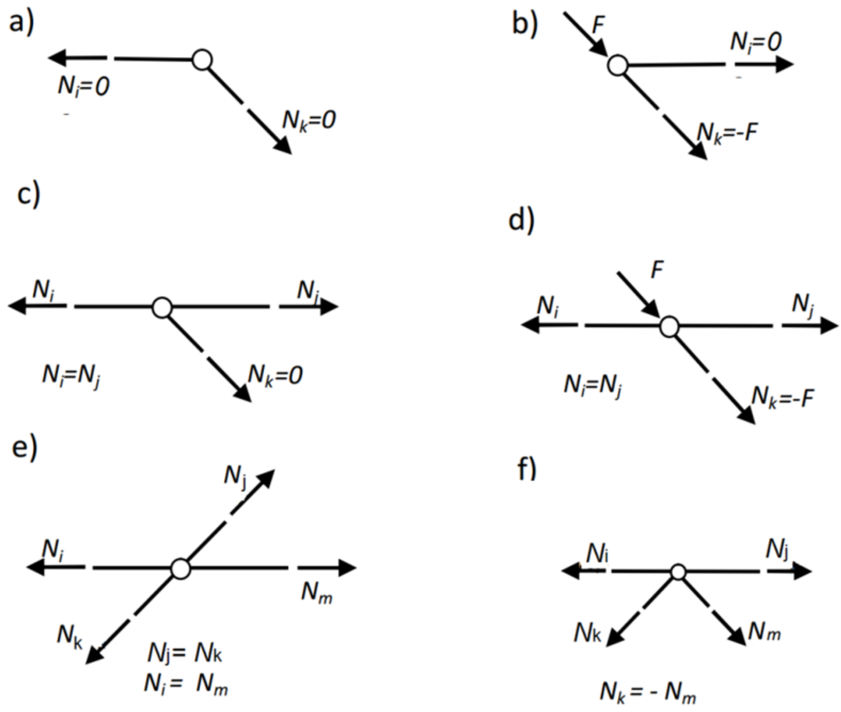


Figure 5.9. Special cases of equilibrium of nodes

We invite the reader to determine unloaded rods in the truss depicted in Figure 5.6, a.

Efforts in some truss rods cannot calculate always immediately, from one equation. Sometimes you have to perform several cuts and draw up the appropriate number of equations. An example of such a bar is the central vertical rod of a trapezoidal truss (Figure 5.7).

To find the force in this rod, you must first find the force in one of the adjacent rods of the upper chord (the first cross section and the first equation) using the moment point method. We suggest doing it yourself. Then cut out (the second cut) the central node of the upper chord (Figure 5.10) and draw up two more equations.

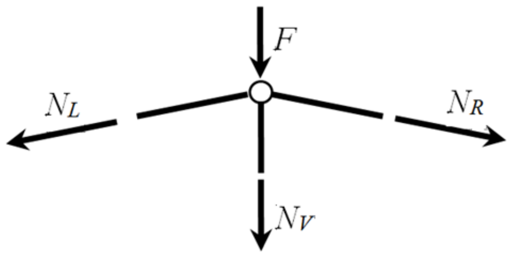


Figure 5.10

The second equation:

$$\Sigma X = -N_L \cos\theta + N_R \cos\theta = 0.$$

Where should

$$N_L = N_R = N.$$

The value of N with its sign is already found from the first equation.
The third equation:

$$\Sigma Y = -2N \sin\theta - F - N_V = 0.$$

Where should

$$N_V = -2N \sin\theta - F.$$

Thus, the force in the central vertical bar is expressed through the extern nodal force and forces in the adjacent rods of the upper chord by the method of two sections.

5.5. Constructing Influence Lines for Internal Forces in the Truss Rods

The influence lines for internal forces in the rods of the beam trusses (Figure 5.11, a) are constructed, as a rule, by the method of sections.

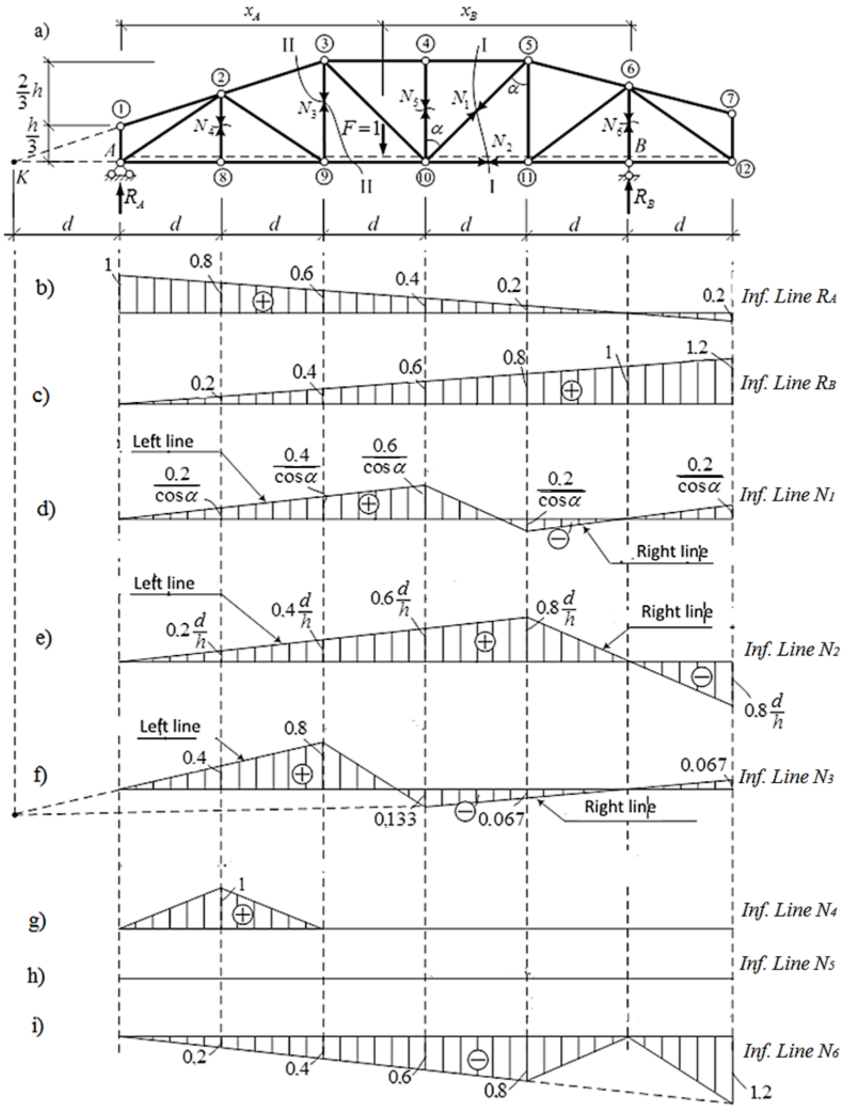


Figure 5.11

First, we construct the influence lines for support reactions. As in a simple beam, to determine the support reactions of the beam truss, we compose the equilibrium equations:

$$\Sigma M_B = 0; \quad -1x_B + R_A 5d = 0; \quad R_A = \frac{x_B}{5d}.$$

$$\Sigma M_A = 0; \quad -R_B 5d + 1x_A = 0; \quad R_B = \frac{x_A}{5d}.$$

The resulting expressions for R_A and R_B are functions of independent variables, respectively x_B and x_A. Their graphs are shown in Figures 5.11, b and 5.11, c.

Thus, the influence lines for support reactions in a beam truss are constructed in exactly the same way as in the corresponding simple beam. Moreover, the influence lines for the support reactions do not depend on which chord the load moves: lower or upper. The efforts in the rods of the truss, as will be shown below, depend on which chord of the truss is loaded: top or bottom.

Assuming for definiteness the movement of a unit force along the lower chord, we will construct an influence line for the internal force N_1 of the diagonal of the fourth panel (the order of consideration of the rods and the construction of force influence lines for them may be arbitrary).

To determine the force N_1, we make section *I–I* and use the sum of the projections of the forces on the vertical axis as the equilibrium equation of one of parts of the truss. This will eliminate unknown forces in the cut up horizontal rods of the lower and upper chords from the equilibrium equation. The equation of the projections on the vertical axis will include only vertical and inclined forces: unit force, support reactions, and the force in the diagonal bar. In the considering case, the mobile force may be on the left part of the truss, or on the right part. Let's consider the possible options.

$$\sum Y^{right} = -N_1 \cos\alpha + R_B = 0;$$

$$N_1 = \frac{R_B}{\cos\alpha}; \qquad I.\,L.\ N_1 = \left(I.\,L.\ R_B\right)/\cos\alpha.$$

That is, the influence line for the effort N_1 in section *A–10* will have the same form as the influence line for the support reaction R_B, all of whose ordinates are divided by $\cos\alpha$ (the angle α is determined from the geometry of the system).

When a unit force moves only to the right of the dissected panel (in the section 11–12), we consider the equilibrium of the left part of the truss:

$$\sum Y^{left} = +N_1 \cos\alpha + R_A = 0;$$

$$N_1 = \frac{R_A}{\cos\alpha}; \qquad I.L.\ N_1 = -\left(I.L.\ R_A\right)/\cos\alpha.$$

We received that the influence line for the effort N_1 in section 11–12 will have the form of a support reaction R_A, all of whose ordinates must be divided by $\cos\alpha$.

When a unit force moves in a section of a dissected panel (in a section of 10–11), the forces in the truss rods in accordance with the principle of nodal transfer of load will change according to a linear law. Therefore, to construct it on this section of the influence line under consideration, it is enough to connect the ordinates of the influence line to the left and right of the dissected panel with a straight line. This straight line segment is called the transition line.

The final form of the influence line for the force N_1 is presented in Figure 5.11, d.

To determine the force N_2 in the rod 10–11 of the lower chord, we can use the same section *I–I* (Figure 5.11, a) and the method of the moment point. We select the moment point in node 5, where the cut rods 4–5 and 10–5 intersect.

Assuming that the unit force is located to the left of the dissected panel 10–11, we consider the equilibrium of the right part of the truss:

$$\sum M_5^{right} = 0; \qquad N_2 h - R_B d = 0;$$

$$N_2 = R_B \frac{d}{h}; \qquad I.L.\ N_2 = \left(I.L.\ R_B\right) \frac{d}{h}.$$

When the unit force is located to the right of the dissected panel 10–11, we consider the equilibrium of the left part of the truss:

$$\sum M_5^{left} = 0; \qquad -N_2 h + R_A 4d = 0;$$

$$N_2 = R_A \frac{4d}{h}; \qquad I.L.\ N_2 = \left(I.L.\ R_A\right) \frac{4d}{h}.$$

In the segment of the dissected panel 10–11, we draw a transition line.

The influence line for the force N_2 is shown in Figure 5.11, e.

An analysis of this influence line shows that its left and right lines intersect under the moment point. This pattern will be satisfied when using the method of the moment point in other cases.

To determine the force in the rod 3–9, we will make section *II–II* (Figure 5.11, a) and also will use the method of the moment point, for which we take the point K of the intersection of the axes of the rods 2–3 and 9–10. The panels of the lower and upper chords dissected by section *II–II* are located on different verticals. In such cases, the position of the movable force should be determined relative to the dissected panels of the loaded chord, i.e., the chord along which the unit force moves. In this case, when the force moves along bottom chord, the dissected panel of the loaded (lower) chord is between nodes 9 and 10. If the unit force moves along top chord, then the upper chord will be loaded, and the dissected panel will be between nodes 2 and 3.

Consider the movement of a unit force along bottom chord.

If a unit force moves to the left of the dissected panel (in the segment A – 9), then, as before, we consider the equilibrium of the right part of the truss:

$$\sum M_\kappa^{right} = 0; \qquad N_3 3d - R_B 6d = 0;$$

$$N_3 = R_B \frac{6d}{3d} = 2R_B; \qquad I.L. N_3 = \left(I.L.\ R_B\right) \cdot 2.$$

When a unit force moves to the right of the dissected panel of the loaded chord (in the area between nodes 10 and 12), we consider the equilibrium of the left part of the truss:

$$\sum M_\kappa^{left} = 0; \qquad -N_3 3d - R_A d = 0;$$

$$N_3 = -0.333 R_A; \qquad I.L.\ N_3 = \left(I.L.\ R_A\right) \cdot \left(-0.333\right).$$

On the length of the dissected panel of the loaded chords (9–10), we draw a transition line. The influence line for the force N_3 is shown in Figure 5.11, f. Its left and right branches intersect under the moment point K.

When constructing the influence line for the force N_4 in the rod 2–8, the force N_4 is determined by cutting out the node 8 (Figure 5.11, a). Node 8 is located on the lower chord of the truss, along which a unit force moves. There are three options for the location of the unit force in relation to the node 8.

1. The unit force is located directly in the cut out node 8. There is a special case (Figure 5.12). The considered rod is stretched by a single force, and the force in it $N_4 = 1.$ We postpone the unit with the plus sign on the influence line under the node.

2. When the unit force is located outside the cut out node, to the left or right of the cut panels of the lower chord, or in any of the nodes of the upper chord, we also have a special case of equilibrium of the node 8 (Figure 5.13), and the internal force $N_4 = 0.$ The ordinates of the influence line are zeros in the corresponding segments.

3. When a unit force moves within the dissected panels of the lower, loaded chord, a nodal load transfer takes place, and in the corresponding segments of the influence line (A – 8 and 8–9), it is necessary to draw transitional lines

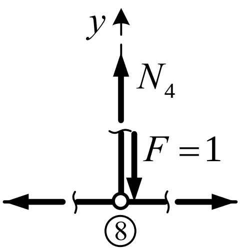


Figure 5.12

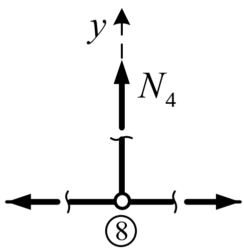


Figure 5.13

The final influence line for the force N_4 has the form shown in Figure 5.11, g.

The effort N_5 in the rod 4–10 is easiest to determine by cutting out the node 4. Again we have a special case of the equilibrium of the node 4.

For any position of the unit force on the loaded lower chord of the truss the effort $N_5 = 0$. Accordingly, the influence line for this effort along the entire length of the truss will be zero (Figure 5.11, h), but only if a unit force moves along the lower chord.

We offer the reader to independently build an influence line for the effort N_5 during the movement of a unit force along the upper chord by himself.

Consider the process of constructing the influence line for the effort in the support column 6 – B. The internal force N_6 in this rod may be easy determined by cutting out the support node B. The equilibrium of the node B is also a special case. The internal force N_6 in the support strut balances the support reaction R_B, only if there is no any other force in this node.

When the unit force is in the cut out support node B (Figure 5.14), it follows from the sum of the projections of the forces on the vertical axis

$$\Sigma Y = N_6 + R_B - 1 = 0$$

that

$$N_6 = 1 - R_B.$$

Thus, the internal force in the support column is equal to minus the support reaction if there is no movable force in the support node. If the movable force is in the support node, then the internal force in the support column is zero. The load is transferred into the support and the whole truss doesn't work.

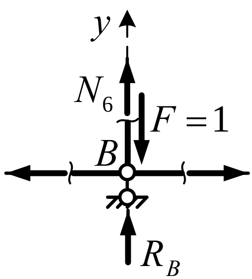


Figure 5.14

The influence line for the effort N_6 repeats the influence line for the support reaction R_B with the minus sign, when the unit force is in all nodes of the truss, except the support node B. When the unit force is in node B, the ordinate of the influence line for the force is equal to zero. Transitional straight lines are in sections of two cut panels of the lower chord.

The final influence line for the effort N_6 is presented in Figure 5.11, i.

5.6. Constructing Influence Lines for Efforts in the Rods of Compound Trusses with Subdivided Panels

Trusses with subdivided panels are formed by superimposing additional secondary trusses on the main truss with a simple lattice. The secondary trusses are located within the panels of the main truss.

The secondary trusses are used to perceive the local load applied between the nodes of the main truss, and transfer it to the nodes of the main truss. Examples of such trusses are shown in Figures 5.15, a and 5.16, a. The decomposition of these compound trusses into the main trusses and secondary ones is shown for these examples in Figures 5.15, b, c and 5.16, b, respectively.

The compound trusses with subdivided panels can be single-tier and double-tier. Single-tier trusses transfer the load to adjacent nodes of the same load chord. For example, for the truss in Figure 5.15, the secondary truss 3-5-6-7 transfers concentrated force from node 5 equally to nodes 3 and 7 of the main truss (Figures 1, b, c).

Double-tier compound trusses, perceiving the load in additional nodes of one chord, transfer it to the main nodes of another chord of the truss. For example, the secondary truss 13-15-17-16-14-18 (Figure 5.16, b) transfers the concentrated force acting in the node 15 of the lower chord to the nodes 14 and 18 of the upper chord.

Three types of rods are distinguished in compound trusses: rods only of the main truss (first type), rods of only secondary trusses (second type) and rods obtained by superimposing secondary trusses rods on the rods of the main truss (third type).

Calculation of compound trusses is carried out, as a rule, by the method of sections. Sometimes it is more convenient to determine the forces in the rods of compound trusses taking into account the belonging of the rods to one of the types listed above. In this case, the calculation sequence is reduced to the following actions.

1. The forces N^S in the secondary truss rods caused by local loads acting on them are determined. The resulting efforts in the rods of the second type are final.

2. The load acting on the secondary trusses is transmitted to the nodes of the main truss. The forces N^M in the rods of the main truss are determined. The efforts obtained in the rods of the first type are final.

3. The forces N in the rods of the third type are calculated by the expression: $N = N^M + N^S$.

We give examples of constructing influence lines for forces in the rods of compound trusses with subdivided panels.

E x a m p l e 1. Consider a beam-type single-tier compound truss (Figure 5.15, a). Let the lower chord of the truss be the loaded chord. To construct the influence lines for the efforts in the rods, one should be guided by the rules for determining the internal forces in compound trusses described above.

a) Let us construct the influence line for the force N_1 in the rod 13-14 (Figure 5.15, a). This rod refers to the rods of the second type, i.e. $N_1 = N_1^S$. To determine the force in it, we use the method of sections in its particular form – the method of cutting out nodes. We cut out the node 13 and consider its equilibrium at various positions of the unit force. When a unit force moves outside the considered node (from node 1 to node 11 and from node 15 to node 21), force N_1^s = 0 (a special case of the equilibrium of the node). If the unit force is in the considered node, then the force N_1^S = 1 (a special case of the equilibrium of the node). The influence line for N_1 is shown in Figure 5.15, d. The indicated influence line is within the truss panel where the secondary truss 11-13-14-15 is located. It means that this secondary truss works only by local loading.

b) The rod 3-5 refers to rods of the third type, i.e. its effort can be found by the expression:

$$N_2 = N_2^S + N_2^M .$$

Therefore, the influence line can be obtained by summing the two influence lines:

$$inf.line\ N_2 = \left(inf.line\ N_2^S\right) + \left(inf.line\ N_2^M\right).$$

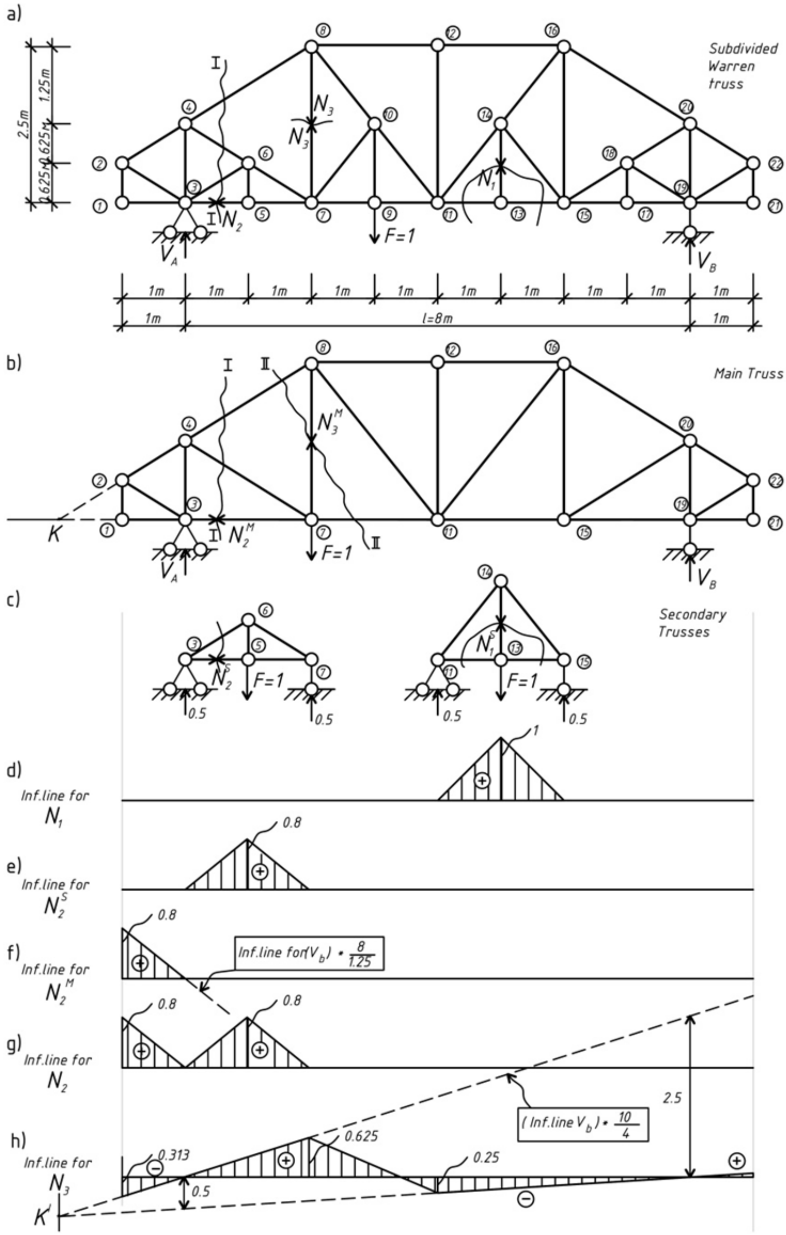

a)
Subdivided Warren truss
2.5m
1.25m
0.625m
0.625m
N_3
N_1
N_2
F=1
V_A
V_B
1m
l=8m
b)
Main Truss
N_3^M
N_2^M
K
c)
Secondary Trusses
N_2^S
N_1^S
0.5
d)
Inf.line for N_1
1
e)
Inf.line for N_2^S
0.8
f)
Inf.line for N_2^M
Inf.line for$(V_b) \cdot \frac{8}{1.25}$
g)
Inf.line for N_2
h)
Inf.line for N_3
0.313
0.625
0.25
0.5
2.5
$(\text{Inf.line } V_b) \cdot \frac{10}{4}$
K'

Figure 5.15

The influence line for N_2^S is construct for the secondary truss 3-5-6-7, therefore, does not go beyond the panel on which the secondary truss is "hung". The rod only works if the moving force locates in the node 5 (Figure 5.15, c). To determine the effort N_2^S we cut out the node 3 of the secondary truss and write the equilibrium equations:

$$\begin{cases}\sum X = 0, \\ \sum Y = 0.\end{cases} \rightarrow \begin{cases}N_{3-6} \cdot \cos\alpha + N_{3-5} = 0, \\ 0,5 + N_{3-6} \cdot \sin\alpha = 0.\end{cases} \rightarrow$$

$$\rightarrow \begin{cases}N_{3-6} \cdot 0.85 + N_{3-5} = 0, \\ 0.5 + N_{3-6} \cdot 0.53 = 0.\end{cases} \rightarrow \begin{cases}N_{3-5} = 0.8; \\ N_{3-6} = -0.94.\end{cases}$$

The influence line for N_2^S is shown in Figure 5.15, e.

The influence line for N_2^M is constructed for the rod 3-7 of the main truss. We draw section *I–I* (Figure 5.15, b). Having compiled the equilibrium equations of all right or left forces, when the moving force is located to the left or to the right of the dissected panel of the load chord, we obtain the dependences for constructing the influence line N_2^M (Figure 5.15, f):

$$\sum M_4^{right} = 0, \rightarrow -V_B \cdot 8 + N_2^o \cdot 1.25 = 0 \rightarrow N_2^o = V_B \cdot \frac{8}{1.25} \rightarrow$$

$$\inf.lineN_2^o = \left(\inf.lineV_B\right) \cdot \frac{8}{1.25}.$$

$$\sum M_4^{left} = 0, \rightarrow V_A \cdot 0 - N_2^o \cdot 1.25 = 0 \rightarrow N_2^o = 0 \rightarrow \inf.lineN_2^o = 0.$$

The influence line N_2 obtained by summing is shown in Figure 5.15, g.

c) Rod 7-8 refers to rods of the first type, i.e. an effort $N_3 = N_3^M$. To construct the influence line N_3^M we do the section *II–II* (Figure 1, b). We obtain expressions for the "left-hand branch" of the influence line N_3^M (Figure 5.15, h):

$$\sum M_K^{right} = 0, \rightarrow -V_B \cdot 10 + N_3^o \cdot 4 = 0 \rightarrow N_2^o = V_B \cdot \frac{10}{4} \rightarrow$$

$$\inf.line N_2^o = \left(\inf.line V_B\right) \cdot \frac{10}{4}.$$

The "right-hand branch" of the influence line N_3^M can be constructed without writing an analytical expression. The "left / right-hand branches" of the influence line for efforts in the rods of trusses of beam type have the properties: to pass through the left / right-hand supports; intersect under the moment point (moment point method) or be parallel in the case of the projection method. "Right-hand branch" of the influence line N_3^M therefore, passes through the support *B* and crosses the "left-hand branch" under the point *K* (Figure 5.15, h). The transition line will connect the left-hand and right-hand branches within the dissected panel of the lower (load) chord.

E x a m p l e 2. Consider a beam-type compound truss with two-tier secondary trusses (Figure 5.16, a). Let the lower chord be the load chord.

a) Let us construct the influence line for the force N_1 in the rod 16-18 (Figure 5.16, a). This rod refers to the secondary truss rods 13-15-17-16-14-18, i.e. $N_1 = N_1^s$. The secondary truss under load is shown in Figure 5.16, b. From the equilibrium of the node 18 of the secondary truss (Figure 5.16, b):

$$\sum Y = 0 \quad \rightarrow \quad 0.5 - N_1^s \cdot \sin 45^\circ = 0,$$

find the effort $N_1^s = 0.707$.

If the unit force moves outside the fourth panel, then the considered secondary truss will not work, and therefore the effort $N_1^s = 0$ (a special case of the equilibrium of the node). The influence line for N_1 is shown in Figure 5.16, c.

b) The rod 14-18 of the same panel refers to the rods of the third type, i.e. the influence line for N_2 is constructed by the expression:

$$inf.line\ N_2 = \left(inf.line\ N_2^S\right) + \left(inf.line\ N_2^M\right).$$

To determine the effort N_2^s we consider the equilibrium of the secondary truss node 18 (Figure 5.16, b):

$$\sum X = 0 \quad \rightarrow \quad -N_2^s - N_1^s \cdot \cos 45^\circ = 0,$$

find the effort $N_2^S = -0.5$.

The influence line for N_2^s is shown in Figure 5.16, d.

The influence line N_2^M is constructed for the rod 14-18 of the main truss. Hold the section *I–I* (Figure 5.16, b). The dependence for constructing the "left-hand branch "of the influence line N_2^M (Figure 5.16, e) is obtained by writing the equation of moments of all right forces relative to the moment point – node 17:

$$\sum M_{17}^{right} = 0, \rightarrow -V_B \cdot 4 - N_2^M \cdot 2 = 0 \rightarrow N_2^M = -V_B \cdot 2 \rightarrow$$
$$\inf .lineN_2^M = -\left(\inf .lineV_B\right) \cdot 2.$$

Right-hand branch" of the influence line N_2^M (Figure 5.16, e) passes through support *B* and crosses the "left-hand branch" under node 17 (Figure 15.16, h). The transition line will connect the left-hand and right-hand branches within the fourth (dissected) panel. The influence line N_2 obtained by summing is shown in Figure 5.16, f.

c) It may seem that the rod 13-14 refers to the rods of the first type, i.e. an effort $N_3 = N_3^M$. In this case the influence line N_3^M is shown in Figure 5.16, g by the dashed line: the upper dashed line is constructed under the condition that the unit force moves along the lower chord; the lower dashed line is constructed under the condition that a unit force moves along the upper chord (in this case, rod 13-14 does not work, because the concentrated force does not fall into node 13).

However, in fact this rod is the support rod (suspension) of two adjacent two-tier secondary trusses located throughout 4 panels between nodes 9 and 17. Consider the location of the force at nodes 11, 13, 15. When the unit force is in the node 13 the internal force N_3 in the rod 13-14 is equal to 1.

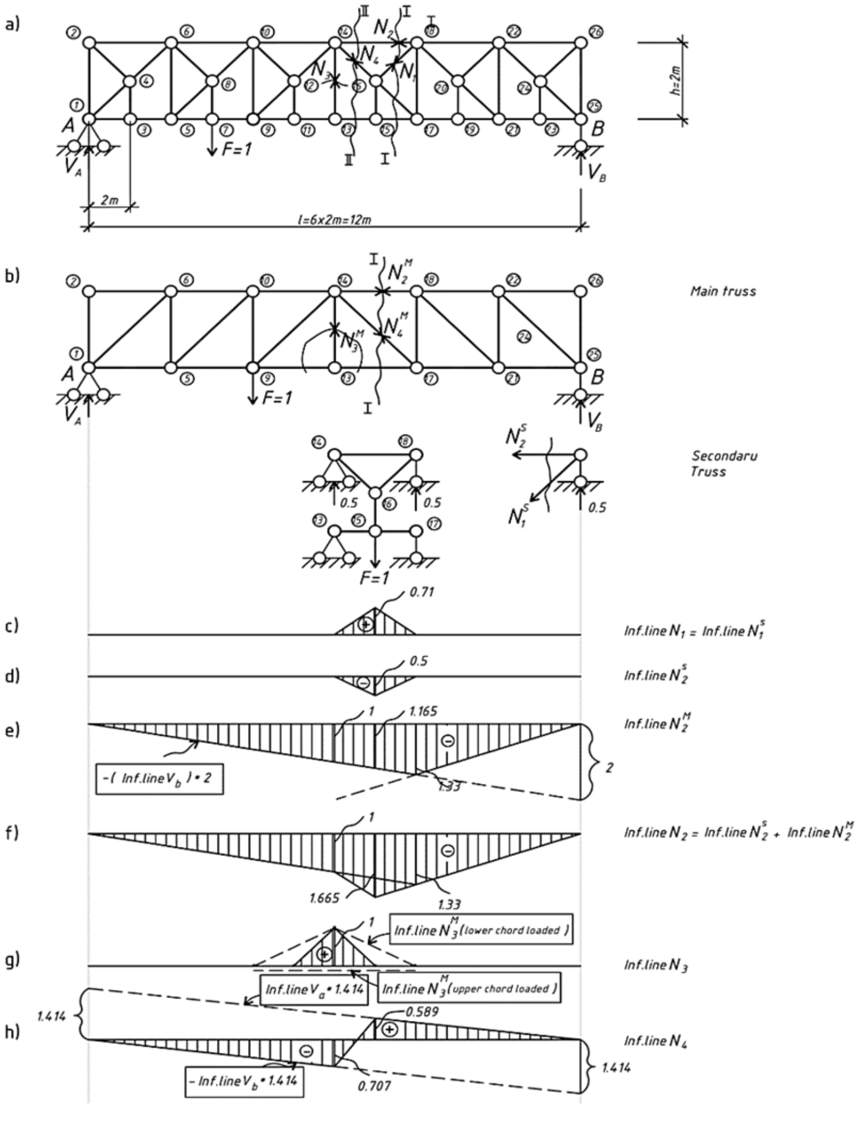


Figure 5.16

Therefore, under the node 13 the ordinate of the upper dashed line will be valid. If the force is located in nodes 11 and 15, the corresponding secondary trusses are included in the work and redistribute force pressure on the upper chord. The node 13 is not loaded and the internal force N_3 is zero. Consequently, the corresponding ordinates of the lower dashed line will be valid.

We show the final form of the influence line for the force N_3 by the shaded part of the Figure 5.16, g.

d) Using similar actions, we will build the influence line for N_4 in the rod 14-16.

Section *II–II* (Figure 5.16, a) passes through the considered rod 14-16, but does not intersect more than three rods with unknown forces. Therefore, it is possible to determine the forces and construct influence lines for the rod 14-16 guided by the rules for determining the internal forces in the truss rods with a triangular or diagonal lattice.

$$\sum Y^{right} = 0, \to V_B + N_4^M \cdot \sin 45^\circ = 0 \to N_4^M = -V_B \cdot \frac{1}{0.707} \to$$

$$\inf.lineN_2^M = -(\inf.lineV_B) \cdot 1.414.$$

$$\sum Y^{left} = 0, \to V_A - N_4^M \cdot \sin 45^\circ = 0 \to N_4^M = V_A \cdot \frac{1}{0.707} \to$$

$$\inf.lineN_2^o = (\inf.lineV_A) \cdot 1.414.$$

In Figure 5.16, h, the left-hand and the right-hand branches of influence line for N_4 are shown in dashed lines. When the load moves to the left-hand of the dissected panel (to the left-hand of node 13), the expression for the "left-hand branch" is valid; when the load moves to the right-hand of the dissected panel (to the right-hand of node 15), the expression for the "right-hand branch" is valid. The transfer line is located within the dissected panel (rod 13-15 of the lower (loaded) chord). The final form of the influence line for N_4 is shown in Figure 5.16, h (the shaded part).

THEME 6. CALCULATING THREE-HINGED ARCHED TRUSSES, COMBINED AND SUSPENSION SYSTEMS

6.1. Calculation of Arched Trusses

In three-hinged arched trusses, unlike three-hinged arches and frames, the system disks consist of hinged-rod systems, i.e. trusses.

In arched trusses not only vertical, but also horizontal components of the support reactions occur under the action of only vertical loads. The horizontal components are called thrust. Examples of arched trusses, trusses with thrust, are shown below (Figures 6.1 and 6.2, a).

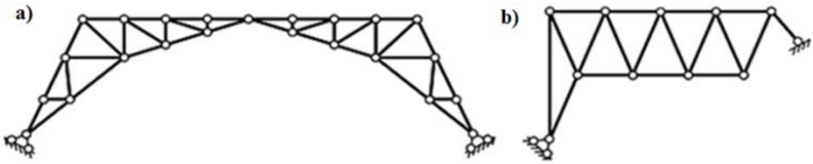


Figure 6.1

The trusses shown in the drawings (Figures 6.1, a and 6.2, a) are called arched trusses, since the method of their formation is similar to the method of forming three-hinged arches. The beam truss (Figure 6.1, b) with an inclined support rod is also a thrusting system, a truss that has thrust under vertical loads.

The support reactions of arched trusses are defined in the same way as in three-hinged arches and frames. After determining the support reactions, the internal forces in the rods of the arched trusses from the action of any load are determined by the same methods as in beam trusses.

Consider the features of constructing influence lines for efforts in the rods of arched trusses. Let's build, for example, an influence line for the internal force N_1 in the rod of the upper chord of the truss (Figure 6.2, a). Before this, it is necessary to construct influence lines for the reactions of the truss.

The vertical component of the reaction of the immovable hinged support A is determined from the equation of moments of all the forces acting on the truss, relative to point B:

$$\Sigma M_B = 0; \qquad R_A l - 1(l - x_F) = 0; \qquad R_A = 1 - \frac{x_F}{l}.$$

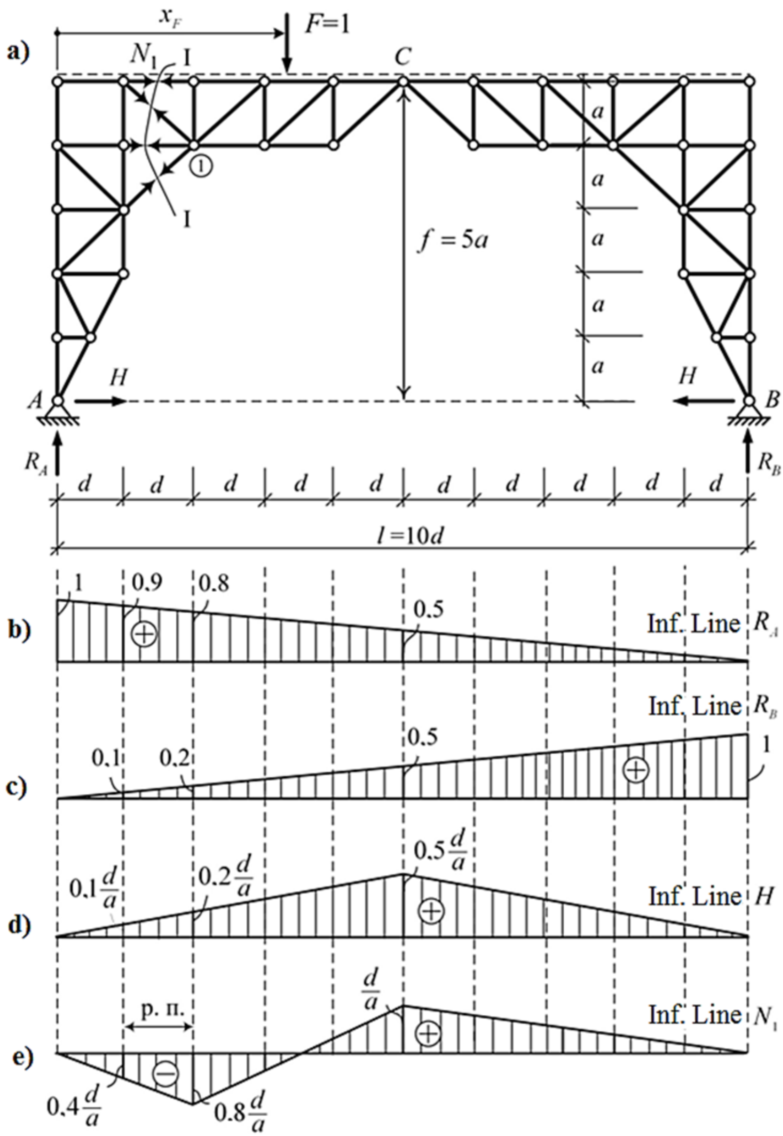


Figure 6.2

The obtained dependence coincides with the corresponding dependence of a simple beam of a span of *l*. Therefore, the influence line of the support reaction R_A is constructed as in a simple beam (Figure 6.2, b).

Similarly, we obtain the dependence for the vertical component of the reaction of the support B:

$$R_B = \frac{x_F}{l}.$$

The influence line for the reaction R_B is shown in Figure 6.2, c.

The horizontal component of the support reactions, i.e. thrust, may be defined, as in a three-hinged arch, according to the formula:

$$H = \frac{M_C^0}{f}.$$

Therefore, the influence line for the thrust is the influence line for the beam bending moment in the beam cross section located under the intermediate hinge of the arch truss, taken with a coefficient $1/f$ (Figure 6.2, d).

The internal force N_1 may be calculated using section *I–I* and moment point 1 (Figure 6.2, a).

If the unit force is located to the left of the section, then, considering the equilibrium of the right part of the truss, we get:

$$\sum M_1^{right} = 0; \qquad -R_B 8d + H4a - N_1 a = 0;$$

$$N_1 = 4H - \frac{8d}{a} R_B.$$

It means that

$$Inf.Line\ N_1 = (Inf.Line\ H) \cdot 4 - (Inf.Line\ R_B) \cdot \frac{8d}{a}.$$

When the unit force moves to the right of the dissected panel, from the equation of equilibrium of the left forces we find:

$$\sum M_1^{left} = 0; \qquad R_A 2d - H4a + N_1 a = 0; \qquad N_1 = 4H - \frac{2d}{a} R_A.$$

It means that

$$Inf.Line\ N_1 = (Inf.Line\ H) \cdot 4 - (Inf.Line\ R_A) \cdot \frac{2d}{a}.$$

In the length of the dissected panel, we draw a transition line. The influence line for the internal force N_1 is shown in Figure 6.2, d.

6.2. Calculation of Combined Systems

Structural systems, some of the elements of which work on bending, shear and tension-compression, and the other part only on tension-compression, are called combined systems. Such systems, for example, include: a beam with a hinged arch (Figure 6.3, a), three-hinged systems (arches¸ frames) with ties of various kinds (Figures 6.3, b, c, d), a beam with a hinged chain (Figure 6.4, a), a suspension hinged chain with a stiffening beam (Figure 6.5) and many others.

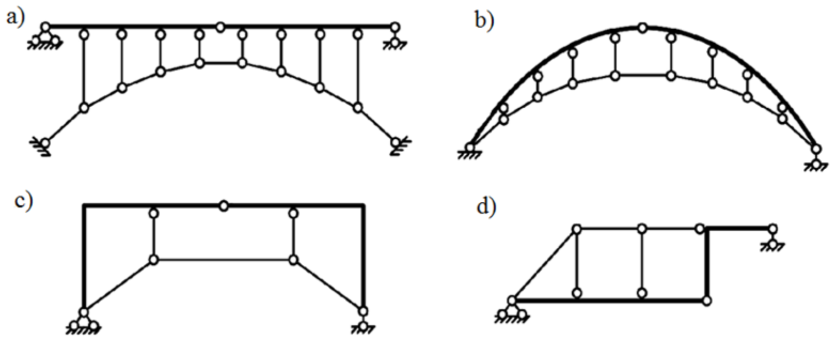


Figure 6.3

Features of the combined systems calculation will be discussed below on the examples of the calculation of a beam with a hinged chain (Figure 6.4, a) and a suspension system. (Figure 6.5, a)

6.3. Calculation of a beam with a hinged chain

A geometrically unchangeable and statically determinate beam with a hinged chain (Figure 6.4, a) is a structure, where the horizontal bars AC and CB connected by an intermediate hinge are strengthened by a polygonal hinged chain with vertical struts.

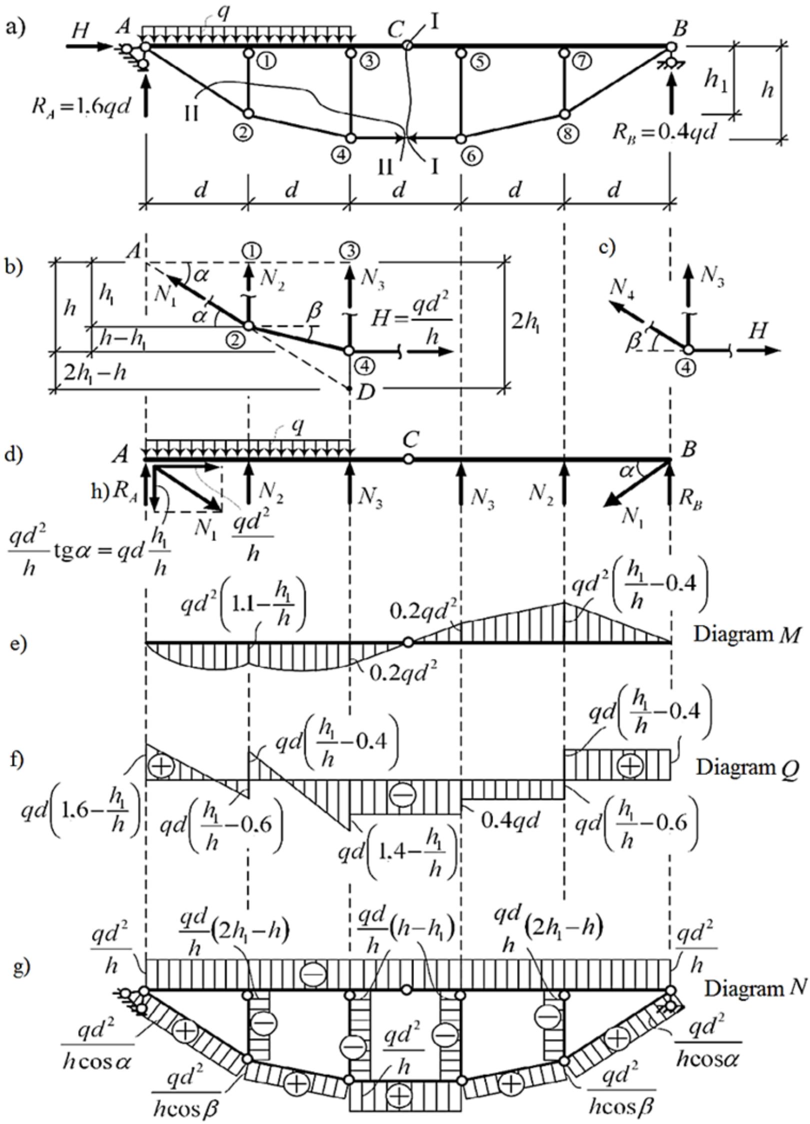


Figure 6.4

The horizontal reaction of support ***A*** is zero under any vertical load.

Vertical support reactions caused by a given load, we find from the equilibrium equations of the entire system:

$$\Sigma M_A = 0; \quad -R_B 5d + q\,2d\,d = 0; \quad R_B = 0.4\,q\,d;$$

$$\Sigma M_B = 0; \quad R_A 5d - q\,2d\,4d = 0; \quad R_A = 1.6\,q\,d.$$

We begin the calculation of internal forces by determining the force H in the rod 4-6 of the hinged chain. To do this, we draw section *I–I* through the named rod and hinge *C*. Considering the equilibrium of the right part, we obtain

$$\sum M_C^{right} = 0; \quad -R_B\,2.5d + Hh = 0; \quad H = R_B \frac{2.5d}{h} = \frac{qd^2}{h}.$$

Then the internal forces in the rods of the chain and in the struts can be determined as in the rods of any truss (Figure 6.4, b).

After determining the forces in the elements of the chain and in the struts, the horizontal bars are calculated on the action of a given load and the forces transmitted by the members of the strengthening system (Figure 6.4, d), like a simple beam. We recommend that the reader perform the corresponding calculations on their own.

Diagrams of internal forces are shown in Figure 6.4, e, f, g.

6.4. Calculation of a suspension system

The features of the influence lines construction for internal forces in the elements of combined systems can be considered using an example of a suspension system such as a hinged chain with a stiffening beam (Figure 6.5, a).

The procedure for determining the forces in the elements of this system is as follows.

To find the support reactions from the action of the load applied to the stiffening beam, the hinged chain must be cut at the points A' and B' located vertically above the supports A and B (Figure 6.5, a). The longitudinal forces in the cut rods can be decomposed into horizontal and vertical components V'_A, H'_A and V'_B, H'_B, Having compiled the equilibrium equations of the lower part of the system in the form of sums of moments relative to points A' and B', the sums of the vertical components $R_A = V_A + V'_A$ and $R_B = V_B + V'_B$ may be found:

$$\Sigma M_{A'} = 0; \quad 1x - R_B l = 0; \quad R_B = \frac{x}{l}; \tag{6.1}$$

$$\Sigma M_{B'} = 0; \quad -1(l - x) + R_A l = 0; \quad R_A = \frac{l - x}{l}. \tag{6.2}$$

From the equations of equilibrium of the hinged chain nodes at the junctions of the vertical suspensions (Figure 6.6, b, c, d) or of a fragment (Figure 6.6, a) it follows that the horizontal component of the longitudinal forces in the chain elements is constant and equal to the thrust of system *H*.

To find the thrust H, we draw section *II–II*, passing through the hinge *C* and the horizontal chain rod (Figure 6.5, a). Having compiled the sum of the moments of forces relative to the hinge C for one of the parts of the system, for example, for the left, we get:

$$\sum M_C^{left} = 0; \qquad R_A \frac{l}{2} - 1\left(\frac{l}{2} - x\right) + Hh - H(h + f) = 0.$$

Or, considering that

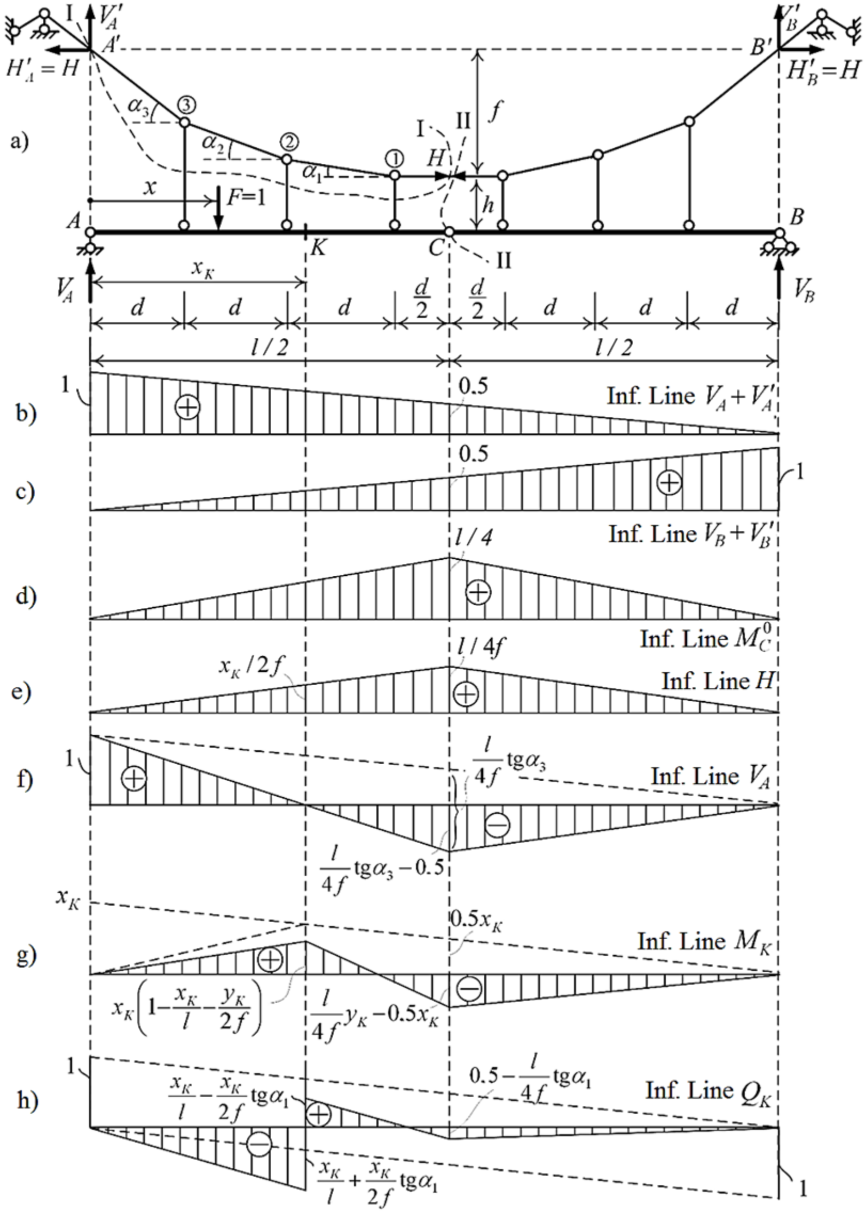
$$R_A \frac{l}{2} - 1\left(\frac{l}{2} - x\right) = M_C^0, \tag{6.3}$$

get the formula for determining the horizontal component *H*

$$H = \frac{M_C^0}{f}. \tag{6.4}$$

From the conditions for the expansion of the longitudinal force in the chain element at the point A' (Figure 6.6, a), we find the vertical component V'_A:

$$V'_A = H \operatorname{tg} \alpha_3.$$



a)
b)
Inf. Line V_A+V_A'
c)
Inf. Line V_B+V_B'
d)
Inf. Line M_C^0
e)
Inf. Line H
f)
Inf. Line V_A
g)
Inf. Line M_K
h)
Inf. Line Q_K

Figure 6.5

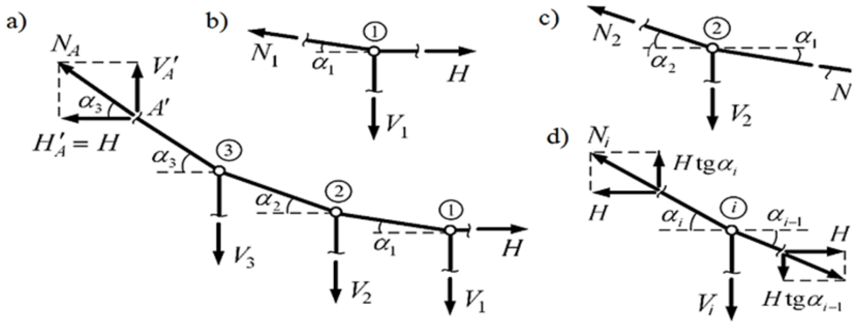


Figure 6.6

Similarly, the component V_B' is determined.

After that we find the support reactions V_A and V_B:

$$V_A = R_A - V_A'; \qquad V_B = R_B - V_B'. \tag{6.5}$$

With the known horizontal component ***H***, the total forces in the chain elements will be equal

$$N_i = \frac{H}{\cos\alpha_i}.$$

Suspension forces are determined from the equilibrium equations of the nodes (Figures 6.6, b, c, d).

To determine the internal forces in the section *K* of the beam, we draw a strictly vertical section through *K* and consider the equilibrium of the left part (Figure 6.7).

We decompose the longitudinal force in the cut chain element into the horizontal and vertical components *H* and V_1. The bending moment and the transverse force in the cross section *K* will be equal to:

$$M_K = (V_A + V_A')x_K - F(x_K - x) - H(h + f) + H(h + f - y_K) =$$
$$= R_A x_K - 1(x_K - x) - Hy_K = M_K^0 - Hy_K, \tag{6.6}$$

$$Q_K = (V_A + V_A') - F - H\ \mathrm{tg}\,\alpha_1 = R_A - 1 - H\ \mathrm{tg}\,\alpha_1 = Q_K^0 - H\ \mathrm{tg}\,\alpha_1, \tag{6.7}$$

where M_K^0 and Q_K^0 are the bending moment and the transverse force in the corresponding section of a simple two-support beam having the same span and the same load as the system under consideration.

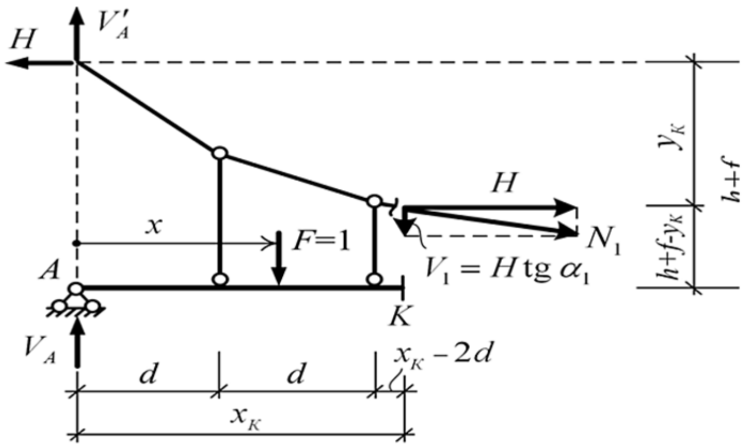


Figure 6.7

Based on the obtained dependencies for determining the support reactions and efforts, it is possible to construct the necessary lines of influence.

So, using formulas (6.1) and (6.2), we build the influence lines $R_A = V_A + V'_A$ (Figure 6.5, b) and $R_B = V_B + V'_B$ (Figure 6.5, c). As for a simple beam, the influence line for the beam bending moment M_C^0 is constructed (Figure 6.5, d). According to the formula (6.4), the influence line for the component H is built (Figure 6.5, e), and on the basis of (6.5) the influence line for the reaction V_A (Figure 6.5, e) is constructed.

According to formulas (6.6) and (6.7), the influence lines of the bending moment M_K (Figure 6.5, g) and the transverse force Q_K (Figure 6.5, h) are plotted.

THEME 7. BASIC THEOREMS OF STRUCTURAL MECHANICS AND DETERMINATION OF DISPLACEMENTS

7.1. Bars Systems Displacements. General Information

When the load is applied to a structure (we will denote this factor by F), when the temperature changes (t) or the supports are displaced (c), linear deflections of its points and the angles of rotation of its cross-sections appear.

In Figure 7.1 the solid line shows the initial state (before the external load applied) of the frame elements, the dashed line shows the state after loading (deformed state). The cross-section K has moved to the position K_1. The angle φ describes the rotation of the cross-section, the section KK_1 (not shown in the diagram) describes the linear displacements of the cross-section K.

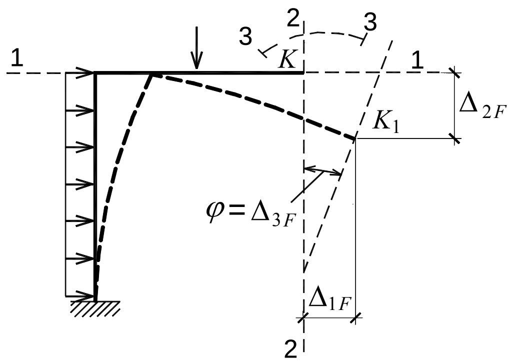


Figure 7.1

The linear displacement of the cross-section K in a direction that does not coincide with the true one can be determined by finding the projection of the segment KK_1 on this direction. In engineering calculations, the displacements of the cross-section in the vertical and horizontal directions are often determined.

The displacements are determined by checking the rigidity of structures, by calculating them for stability and vibrations, and also by calculating statically indeterminate systems.

The displacement of any cross-section is usually denoted with a symbol Δ (delta) with two indices, the first of which indicates the direction of displacement, and the second one indicates the reason that caused the displacement. So, for example, Δ_{1F} is denoted the displacement of the cross-section in the 1st direction, caused by an external load. The sense of the notation Δ_{2F} and Δ_{3F} is revealed with the help of Figure 7.1. Then, it will be necessary to determine the displacements in the direction of several concentrated forces $F_1, F_2, \cdots, F_n$ action. Then Δ_{iF} should be read as follows: this is the displacement of the application point of the force F_i in its direction caused by the load F.

The displacement in the *i*-th direction caused by the temperature effect is denoted as Δ_{it}, the displacement in the *i*-th direction caused by the displacement of the supports is denoted as Δ_{ic}.

Determination of displacements in linearly deformable systems is based on general theorems on elastic systems.

7.2. Work of External Statically Applied Forces

The load on any structure causes the movement of the structure from the initial state to a new, deformed one. We will consider such a load that is applied to the structure so slowly, smoothly, that the resulting accelerations of its elements, and therefore, the inertial forces of their masses can be neglected. The loading process is called static, and the corresponding load is called static.

Let a rod made of a nonlinear elastic material undergo tensile force F (Figure 7.2).

The stress-strain diagram of this material is shown in Figure 7.3.

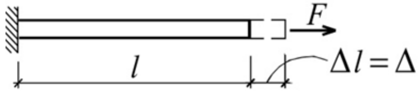


Figure 7.2

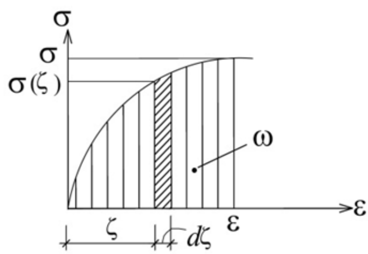


Figure 7.3

The area of the diagram ω, as is known from the course "Strength of materials", is equal to the specific potential energy u_0 (in other words, the energy density is the energy referred to the unit of the initial volume of the element) under a linear stress state.

If we change the scale of the diagram $\sigma - \varepsilon$ ordinates by introducing the dependencies $N = \sigma A$ and $\Delta l = \varepsilon l$, then we can get the dependence "load-displacement" that is often used in the practice of calculations (Figure 7.4).

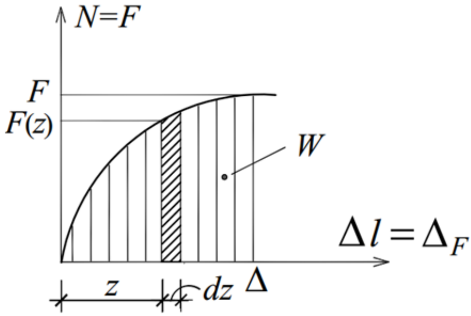


Figure 7.4

In this figure symbol z denotes some intermediate absolute elongation of the rod caused by force $F(z)$, and symbol Δ denotes the displacement corresponding to the final (maximum) value of the force F.

The work performed by force with infinitely small increase in displacement by dz is determined by the expression: $dW = F(z)\,dz$.

Summing up the elementary work over the entire range of displacements change we obtain a formula for determining the work performed by a statically applied external force F:

$$W = \int_0^{\Delta} F(z)\,dz.$$

For a linear-elastic rod, the ratio between force and displacement is linear (Figure 7.5). Therefore, $F(z) = kz$, where k is the stiffness coefficient of the rod.

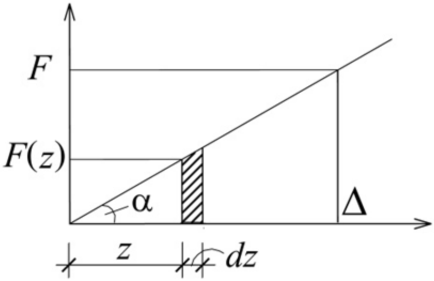


Figure 7.5

The final value F of the force corresponds to displacement Δ. The work of the statically applied force is calculated by the expression:

$$W = \int_0^{\Delta} F(z)\,dz = \int_0^{\Delta} k\,z\,dz = \left.\frac{kz^2}{2}\right|_0^{\Delta} = k\frac{\Delta^2}{2}.$$

Since $k = \mathrm{tg}\alpha = \dfrac{F}{\Delta}$, then $W = \dfrac{F\Delta}{2}$.

The work of an external statically applied force is equal to half the product of the value of this force by the value of the displacement caused by it (Clapeyron`s theorem (1799–1864)). The work of a statically applied force on the displacement caused by the same force is called actual work.

In the general case, by force it is necessary to understand not only concentrated force, but also moment and distributed load. The corresponding displacements will be linear displacement in the direction of the force, angular in the direction of the moment, and the area of the displacement diagram at the action region of the distributed load.

With the mutual action on the system of several statically applied forces, their work is calculated as half the sum of the products of each force on the corresponding total displacement:

$$W = \frac{1}{2}\sum^{i} F_i \Delta_i. \qquad (7.1)$$

For example, with a static action on the beam of concentrated forces F_1, F_2 and of concentrated moment M (Figure 7.6) the actual work of external forces is equal to:

$$W = \frac{F_1 \Delta_1}{2} + \frac{F_2 \Delta_2}{2} - \frac{M\varphi}{2}.$$

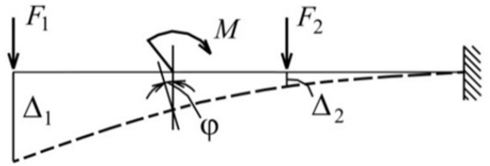


Figure 7.6

The minus sign in the last term of the expression is accepted because the direction of the angle φ of rotation of the cross-section of the beam and the direction of the moment M are opposite.

7.3. Work of the Internal Forces in a Plane Linear-Elastic Bars System

Under the static action of external forces on a deformable system, internal forces arise in its cross-sections. To determine the work of these forces, we cut out an element of length dx (Figure 7.7, a) with the help of infinitely close located cross-sections (Figure 7.7, b).

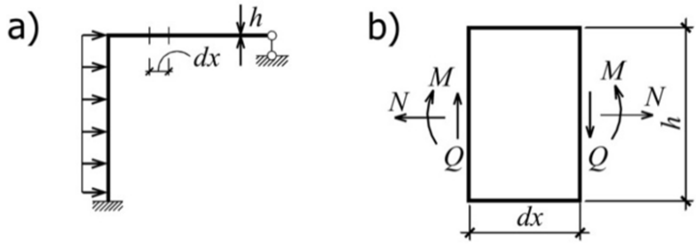


Figure 7.7

With respect to this element, the forces N, M and Q, which replaces the action of the discarded parts of the system on the selected element, are external. Internal forces are equal to them, but opposite in direction. Internal forces are resist element deformations. Therefore, the work of internal forces is always negative.

Note. In the formulas of Section 7.3 and below, the following notation will be used:

A – is an area of the bars cross-section;

J – is an axial moment of inertia of a cross-section; the denote of the moment of inertia J_y in the Zhuravsky`s formula is associated with the axes in Figure 7.9;

EA – is a rigidity of the bar in tension-compression;

EJ – is a bending rigidity of the bar;

GA – is a shear rigidity of the bar.

The impact on the element of longitudinal forces N causes it to stretch by value $\Delta dx = \dfrac{N\,dx}{EA}$ (Figure 7.8, a). On this displacement, a statically rising external force N will perform elementary actual work: $dW_N = \dfrac{1}{2} N\,\Delta dx = \dfrac{N^2\,dx}{2EA}$. The work of the internal longitudinal forces dA_N will be equal to it, but negative (the directions of the internal forces and the corresponding deformations are opposite). Consequently, $dA_N = -dW_N = -\dfrac{N^2\,dx}{2EA}$.

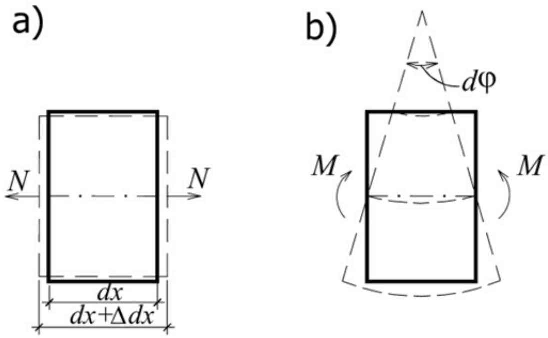


Figure 7.8

At the angular displacement $d\varphi$ of the cross-sections caused by the action of the bending moment $\boldsymbol{M}$ (Figure 7.8, b) its work will be equal to $-\frac{1}{2}M\,d\varphi$.

Using the formula for determining the curvature $\frac{1}{\rho}=-\frac{d^2y}{dx^2}=\frac{M}{EJ}$ of the axis of the bar, the expression of the angle of mutual rotation of the cross-sections can be written in the form $d\varphi=\frac{dx}{\rho}=\frac{M\,dx}{EJ}$. Then $dA_M=-\frac{M^2\,dx}{2EJ}$.

The tangential stresses in the cross-section, determined by the Zhuravsky`s formula:

$$\tau=\frac{Q\,S_y^{cut}}{J_y\,b(z)},$$

cause a mutual shear of the cross-sections $\Delta_Z=\gamma\,dx=\frac{\tau}{G}dx$ (Figure 7.9).

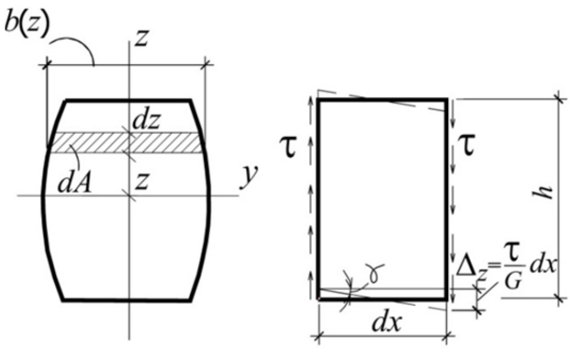


Figure 7.9

To determine their work, we select the corresponding strips with an area dA at the ends of the element dx. Given the static nature of the load, we find that:

$$dA_Q = -\frac{1}{2}\int_A (\tau\, dA)\,\Delta_Z = -\frac{dx}{2\,G}\int_A \tau^2 dA =$$

$$= -\frac{Q^2\,dx}{2\,G}\int_A \left(\frac{S_y^{cut}}{J_y\,b(z)}\right)^2 dA = -\frac{\mu\,Q^2\,dx}{2\,G\,A},$$

where $\mu = A\int_A \left(\frac{S_y^{cut}}{J_y\,b(z)}\right)^2 dA$ – is the dimensionless coefficient depending on the shape of the cross-sectional area.

For a rectangular cross-section μ = 1,2; or round cross-section μ = 1,18; for rolling I-beams approximately μ is equal to the ratio of the area of the I-beam to the area of its wall.

We obtain the full actual work of the internal forces of a plane bars system by integrating the expressions for elementary work along the length of each part of the bar and summing over all parts of the system. The total actual work of internal forces is equal to:

$$A_{\text{int}} = -\sum\int\frac{N^2 dx}{2EA} - \sum\int\frac{M^2 dx}{2EJ} - \sum\int\frac{\mu Q^2 dx}{2GA}. \tag{7.2}$$

Since in the formula (7.2) value N, M and Q are squared, the work of internal forces is always negative.

The relationship between loads and displacements (forces) is linear in linearly deformable systems. The relationship between the load and work, as follows from formula (7.2), is non-linear. The actual work of a group of simultaneously acting external forces is not equal to the sum of the actual works caused by each of the forces individually. The superposition principle of the action of forces in calculating the actual work is not applicable.

7.4. Application of Virtual Displacements Principle to Elastic Systems

We expand the concepts presented in section 2.4.

An elastic system loaded by a given external action takes a definite deformed position. The displacements of the system points counted from the initial (undeformed) state of the system till their corresponding positions in the deformed state are actual displacements.

We set the virtual displacements for the considered system. Since the position of the elastic system in a deformed state is characterized by an infinitely large number of parameters, such a system is a system with an infinitely large number of degrees of freedom. The number of virtual displacements will also be infinitely large.

As noted in section 2.4, while "passing" system from the deformed state to a new, which takes into account the virtual displacements, external actions and internal forces do not change. Therefore, the work of external and internal forces on virtual displacements must be determined by the expressions:

$$W^{(virt)} = \sum F_i \Delta_i,$$

where F_i – generalized forces;

Δ_i – corresponding generalized displacement;

$$A_{\text{int}}^{(virt)} = -\sum S_i e_i,$$

where S_i – generalized internal forces;

e_i – corresponding generalized deformations.

The work of internal forces is always negative.

The formal notation of the principle of virtual displacements is the same as in section 2.4:

$$W^{(virt)} + A_{\text{int}}^{(virt)} = 0.$$

It is assumed that the constraints are ideal in an elastic system, and for virtual displacements, no work is required to overcome friction or to generate and release heat, etc. This is taken into account in inelastic systems.

In practical applications, virtual displacements are the small displacements that can be caused by force actions or other ones. For example, for the beam state shown in Figure 7.10 (state "i"), as virtual displacements one can take the displacements of the same beam loaded with another group of forces (state "k").Then the virtual work of the external forces of the state “i” at the displacements of the state “k” is written in the form:

$$W^{(virt)} = F_1\,\Delta_{1k} + F_2\,\Delta_{2k}.$$

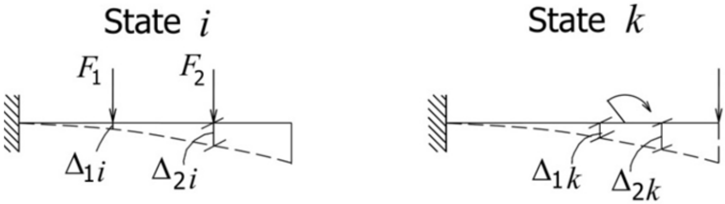


Figure 7.10

The virtual work of the internal forces of the state “i”on the beam deformations in the state “k” will be equal to:

$$A_{\text{int}}^{(virt)} = -\sum\int N_i \frac{N_k\,dx}{EA} - \sum\int M_i \frac{M_k\,dx}{EJ} - \sum\int \mu Q_i \frac{Q_k\,dx}{GA}.$$

The principle of virtual displacements is one of the basic principles of mechanics. It allows one to find equilibrium conditions, which are very important, without determining unknown links reactions.

If actual displacements are taken for virtual displacements, then the virtual work of external and internal forces will be determined by the expressions:

$$W^{(virt)} = \sum^{i} F_i \Delta_i,$$

$$A_{\text{int}}^{(virt)} = -\sum\int \frac{N^2 dx}{EA} - \sum\int \frac{M^2 dx}{EJ} - \sum\int \frac{\mu Q^2 dx}{GA}, \tag{7.3}$$

where $W^{(virt)}$ is virtual work of external forces;

$A_{\text{int}}^{(virt)}$ is virtual work of internal forces.

Note that the concept of the virtual displacement (indicated by a symbol δ) was introduced by Lagrange. In the classical treatise "Analytical Mechanics" (1788; Russian transl., Vols. 1–2, 2 ed., 1950), he considered the "general formula", which is the principle of virtual displacements, as the basis of all statics, and the "general formula", which is a combination of the principle of virtual displacements with the D'Alembert principle, he considered as the basis of all dynamics.

7.5. Theorems of Reciprocity Works and Displacements

Suppose that a linearly deformable system (Figure 7.11, a) is sequentially loaded first with force F_i, and then with force F_k.

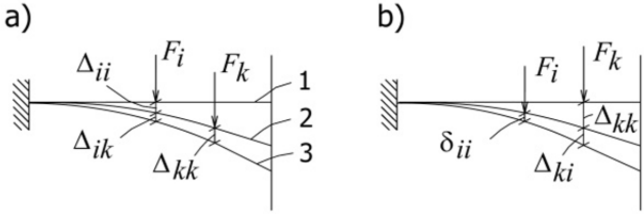


Figure 7.11

When the beam proceeds from position 1 to position 2, then the actual work of the force F_i on the displacement Δ_{ii} is equal to $W_{ii} = \frac{1}{2} F_i \Delta_{ii}$.

When the beam proceeds from position 2 to position 3, then the actual work of the force F_k is equal to $W_{kk} = \frac{1}{2} F_k \Delta_{kk}$, and the force F_i, remaining unchanged at this time, does the virtual work $W_{ik} = F_i \, \Delta_{ik}$ on the displacement Δ_{ik}. The total work of two forces will be equal to:

$$W_1 = W_{ii} + W_{kk} + W_{ik}.$$

If the beam is loaded in the reverse sequence (first by force F_k, and then by force F_i (Figure 7.11, b)), then we obtain:

$$W_2 = W_{kk} + W_{ii} + W_{ki}.$$

Since the value of the work of external forces is equal to the potential energy of the system and, regardless of the loading sequence, in both cases the initial and final positions of the beam coincide, then $W_1 = W_2$. So, we have the equation:

$$W_{ik} = W_{ki}. \tag{7.4}$$

In expanded form:

$$F_i \, \Delta_{ik} = F_k \, \Delta_{ki}.$$

A formal record of the theorem of reciprocity work is obtained (Betty's theorem (1823–1892)): **the work of the forces of the state "*i*" on the displacements of the state "*k*" is equal to the work of the forces of the state "*k*" on the displacements of the state "*i*".**

Note, that in the above formulation, the term "force" should be understood as "generalized force", which can be a group of forces, and the term "displacement" as "generalized displacement".

A similar dependence exists for the virtual work of internal forces on the corresponding deformations. Then the statement of the theorem of reciprocity work can be given in the following form: **the virtual work of the external (internal) forces of the state i on the displacements (deformations) of the state k is equal to the work of the external (internal) forces of the state k on the displacements (deformations) of the state i.**

E x a m p l e. A beam (Figure 7.12) of a constant section in state 1 is loaded with a uniformly distributed load of intensity q, and in state 2 it is loaded with a concentrated moment M applied at the end point. Show the validity of the theorem of reciprocity work.

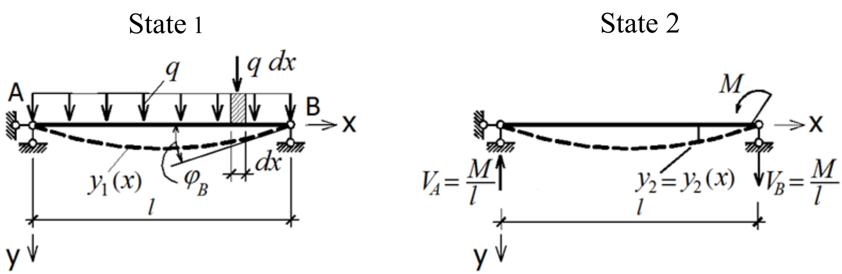


Figure 7.12

The generalized force in state 1 is the load q. Its virtual work is defined as the sum of elementary works of the forces $q\,dx$ on the displacement y_2 of the state 2:

$$W_{12} = \int_0^l q\,dx\;y_2 = q\int_0^l y_2\,dx = q\,\omega,$$

where ω is the area of the diagram of the vertical displacements of the beam in the state 2.

To determine ω we find the equation of the bended axis of the beam. The differential equation of the bended axis is written in the form:

$$EJ\;y_2''(x) = -\frac{M}{l}x.$$

Sequential integrating gives:

$$EJ\;y_2'(x) = -\frac{M}{2l}x^2 + c_1,$$

$$EJ\;y_2(x) = -\frac{M}{6l}x^3 + c_1\,x + c_2.$$

Using the boundary conditions $x = 0$ $y_2 = 0$ and $x = l$ $y_2 = 0$, we find:

$$EJ\;y_2(x) = \frac{M}{6}\left(-\frac{x^3}{l} + l\,x\right).$$

Then:

$$\omega = \int_0^l y_2(x)dx = \frac{M}{6EJ}\int_0^l\left(-\frac{x^3}{l} + l\,x\right)dx = \frac{M\,l^3}{24\,EJ}.$$

The virtual work is:

$$W_{12} = \frac{q\,Ml^3}{24\,EJ}.$$

The virtual work of the concentrated moment M is $W_{21} = M\,\varphi_B$.

Displacements and angles of rotation of the beam in state 1 are determined from the equations:

$$y_1(x) = \frac{1}{EJ}\left[\frac{q\,l^3}{24}x - \frac{q\,l}{12}x^3 + \frac{q\,x^4}{24}\right],$$

$$y_1'(x) = \frac{1}{EJ}\left[\frac{q\,l^3}{24} - \frac{q\,l}{4}x^2 + \frac{q\,x^3}{6}\right].$$

When $x = l$ $\quad y_1'(x) = \varphi_B = -\dfrac{q\,l^3}{24\,EJ}$.

The direction of action of the moment M coincides with the direction of displacement φ_B, therefore:

$$W_{21} = \frac{M\,q\,l^3}{24\,EJ}.$$

Consequently, $W_{12} = W_{21}$.

If the generalized forces in the states "i" and "k" are equal to one (displacements from unit forces are indicated by the symbol δ, Figure 7.13), then it follows from theorem (7.4) that:

$$\delta_{ik} = \delta_{ki}. \tag{7.5}$$

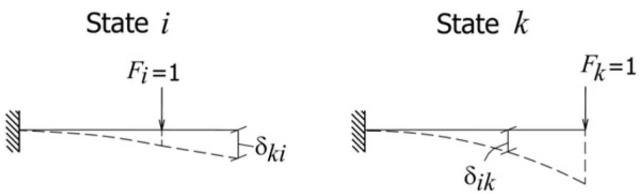


Figure 7.13

Equality (7.5) expresses one of the general properties of linearly deformable systems and is a formal record of the theorem of reciprocity displacements (Maxwell's theorem (1831–1879)): **displacement in the *i*-th direction from the *k*-th unit force is equal to displacement in the *k*-th direction from the *i*-th unit force**.

Remark on the dimension of displacement δ_{ik}. The generalized displacement Δ_{ik}, caused by the generalized force F_k, is defined as $\Delta_{ik} = \delta_{ik} F_k$. Therefore, the dimension of displacement δ_{ik} is obtained in the form:

$$\textit{dimension of } \delta_{ik} = \frac{\textit{dimension of } \Delta_{ik}}{\textit{dimension of } F_k}.$$

For example, when loading the beams shown in Figure 7.14, we have:

$$\delta_{21} = \frac{\Delta_{21}}{F_1}, \quad \textit{dimension of } \delta_{21} = rad/kN = kN^{-1};$$

$$\delta_{12} = \frac{\Delta_{12}}{F_2}, \quad \textit{dimension of } \delta_{12} = m/(kN \cdot m) = kN^{-1}.$$

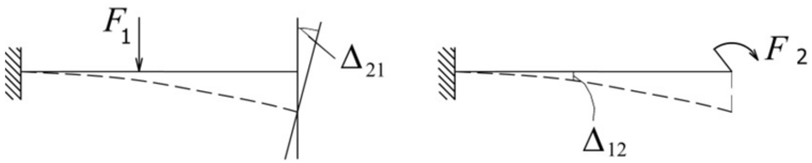


Figure 7.14

Displacements δ_{12} and δ_{21} have the same dimension.

7.6. General Formula for Determining Plane Bars System Displacements

Suppose that the bars system (Figure 7.15, a) has been deformed under the influence of given actions, and it is required to determine the displacement of any of its points *i* in a predetermined direction that does not necessarily coincide with the true direction of displacement of this point.

Considered system state we denote as a "state a", and internal forces in the cross-sections of the elements we denote by N_a, M_a, Q_a. In general, there are elongation $\Delta dx = \varepsilon\, dx$, bending $d\varphi = \kappa\, dx$ and shear $\Delta_z = \gamma\, dx$ deformations in the infinitesimal element of this system in the deformed state. Here, dx is the length of the element, ε is the relative elongation (shortening) of the element, $\kappa = \dfrac{1}{\rho}$ is the curvature of the bended axis, γ – is the relative shear (angle of shear) of the edges of the element.

To determine the required displacement we consider the auxiliary (fictitious) state of the system. In this auxiliary state, we attach a unit generalized force to the same system in the direction of generalized unknown displacement (Figure 7.15, b).

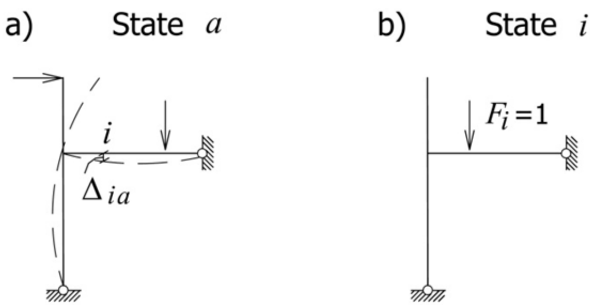


Figure 7.15

The internal forces in this state (state i) of the system are denoted by N_i, M_i, Q_i. Since this state is a state of equilibrium, the principle of virtual displacements can be applied to it. For virtual displacements, we take the displacements caused by a given action. The total work of the external and internal forces of the state i on the displacements of the state a should be equal to zero (7.3), that is:

$$W^{(virt)} + A_{\text{int}}^{(virt)} = 1\,\Delta_{ia} - \sum\int N_i\,\varepsilon\,dx - \sum\int M_i\,\kappa\,dx - \sum\int Q_i\,\gamma\,dx = 0$$

Integration is carried out along the length of each bar or section of the bar, during which the integrand is a continuous function of a certain kind.

Consequently,

$$\Delta_{ia} = \sum\int N_i\,\varepsilon\,dx + \sum\int M_i\,\kappa\,dx + \sum\int Q_i\,\gamma\,dx. \tag{7.6}$$

The obtained formula allows us to express the required displacement through deformations of the system elements in the state a, and the system itself can be both linear and physically nonlinear. The cause of the deformation of the elements is also insignificant: force impact, change in ambient temperature, creep of the material or other reasons. Therefore, formula (7.6) can be considered as a general formula for determining the displacements of bars systems.

The state of the system under the action of a given load is called loaded state (state F). From the course of resistance of materials it is known that the deformations of elements of a linearly deformable system in this state are determined through internal forces as follows:

$$\varepsilon\,dx = \frac{N_F dx}{EA}, \qquad \kappa\,dx = \frac{M_F dx}{EJ}, \qquad \gamma\,dx = \frac{\mu\,Q_F\,dx}{GA},$$

where EA, EJ, GA – the rigidity of the element, respectively, in tension (compression), bending and shear.

Substituting these expressions in (7.6), we obtain a formula for determining the displacements of a plane bar system in the following form:

$$\Delta_{iF} = \sum\int\frac{N_i\,N_F\,dx}{EA} + \sum\int\frac{M_i\,M_F\,dx}{EJ} + \sum\int\frac{\mu\,Q_i\,Q_F\,dx}{GA}. \tag{7.7}$$

This formula is called the Maxwell-Mohr formula for determining the displacements of elastic systems caused by a given load.

The relative contribution of each of the three terms of formula (7.7) to the final result depends on the type of the bars system and the nature of loading. In particular, it appears that the displacements in the beams depend mainly only on the second term (bending moments); the proportion of the term, taking into account the influence of shear forces, is a negligible fraction of the final value Δ_{iF}. Therefore, with sufficient accuracy for practical purposes, the displacements of systems that primarily perceive bending can be calculated by the formula:

$$\Delta_{iF} = \sum \int \frac{M_i M_F \, dx}{EJ}.$$

For the same reason, the calculations (especially "manually") of the frame and arch systems neglect the influence of longitudinal and shear forces in determining displacements. At the same time, the automated calculation of these systems using computer programs is carried out, as a rule, taking into account bending moments and longitudinal forces in determining displacements.

In elements of trusses with hinged joints only longitudinal forces arise from the node loads. Therefore, the determination of the displacements of nodes in the trusses is made according to the formula:

$$\Delta_{iF} = \sum \int_0^l \frac{N_i N_F \, dx}{EA}.$$

Since with a nodal load on the truss, the longitudinal force along the length of the rod does not change, then, provided that the rigidity of each rod is constant, the formula is rewritten in the form:

$$\Delta_{iF} = \sum_{k=1}^{n} \frac{N_{ki} N_{kF} \, l_k}{EA_k}, \tag{7.8}$$

where l_k – the length of the k-th rod;

n – number of truss rods.

In this form (7.8), for the first time in 1864, J. Maxwell obtained a formula for determining the displacements of trusses. 10 years later, O. Mohr (1835–1918) developed a method for determining displacements for the case of arbitrary deformations of the system (see formula (7.7)).

Let us explain the features of the choice of the auxiliary state. A single generalized force must be applied to the system in the direction of the corresponding generalized displacement. Their product, as you know, gives the work of force $F=1$ on the required unknown displacement. If, for example, for the frame in the state F (Figure 7.16, a), it is necessary to determine the angle of rotation of any cross-section of the element, for example, the cross-section D, then in the auxiliary state in this section it is necessary to apply a single concentrated moment $M_1=1$ (Figure 17.16, b), and then the virtual work of the external force in the state *"1"* on the displacement of the state "F" will be equal $M_1\varphi = 1\cdot\Delta_{1F}$. Subsequently, the index of the unit load in the auxiliary state will determine the number of this state.

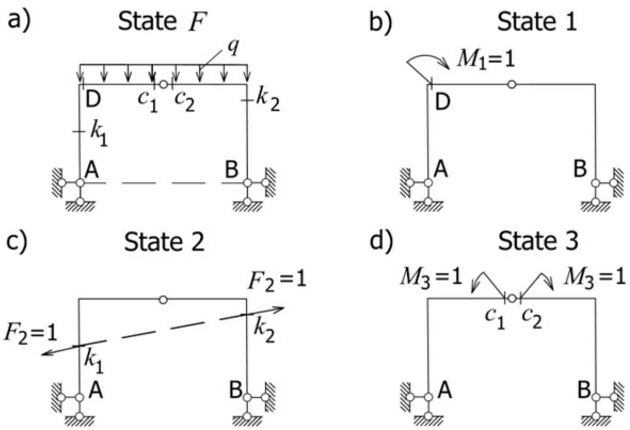


Figure 7.16

If it is necessary to determine the change in the distance between points k_1 and k_2, then in the auxiliary state (state 2) two unit forces directed in opposite sides should be applied along the direction of the line connecting these points (Figure 7.16, c); if it is necessary to find the angle of mutual rotation of the cross-sections c_1 and c_2, then in the auxiliary state (state 3), two opposite-directional moments should be applied in these cross-sections (Figure 7.16, d) each being equal to the unit.

The directions of unit forces given in auxiliary states correspond to positive directions of displacement Δ_{iF}. If the result of the calculation is $\Delta_{iF}<0$, then it will mean that the required displacement is directed in the direction opposite to the direction of the force $F_i=1$.

7.7. Mohr Integrals. Ways for Calculation

The problem of calculating displacements using the Mohr's formula reduces to calculating integrals of the form

$$\int_a^b \frac{M_i M_F \, dx}{EJ},$$

which are commonly called Mohr integrals. For relatively simple problems, the integrant

$$f(x) = \frac{M_i M_F}{EJ}$$

can be such that the indefinite integral $F(x)$ can be expressed using a finite number of elementary functions. Then a definite integral is calculated by the formula

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Let us show, for example, the determination of the vertical displacement of cross-section 1 and the angle of rotation of cross-section 2 of the cantilever beam (Figure 7.17), loaded with a uniformly distributed load. The influence of only bending moments to deflection only we will take into account.

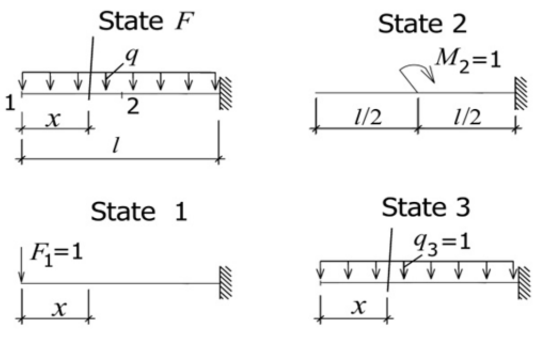


Figure 7.17

To determine the deflection, we use the auxiliary state 1. In the future, the designations of forces from dimensionless forces will be accompanied by the upper line. Then:

$$M_F = -0.5qx^2, \quad \bar{M}_1 = -1\,x.$$

Taking the bending rigidity of the beam EJ constant along its length, we obtain:

$$\Delta_1^{(vert)} = \Delta_{1F} = \int_0^l \frac{\bar{M}_1 M_F\,dx}{EJ} = \int_0^l \frac{1}{EJ}(-x)\,(-\frac{qx^2}{2})\,dx = \frac{q\,l^4}{8EJ}.$$

To determine the angle of rotation of the cross-section in the middle of the beam, we use the auxiliary state 2. Then:

$$M_F = -\frac{q\,x^2}{2};$$

$$\bar{M}_2 = 0\,,\ \text{if}\ 0 \le x < \frac{l}{2};$$

$$\bar{M}_2 = 1,\ \text{if}\ \frac{l}{2} \le x \le l:$$

$$\varphi_2 = \Delta_{2F} = \sum\int \frac{\bar{M}_2 M_F\,dx}{EJ} = \int_0^{\frac{l}{2}} \frac{1}{EJ}\,0\,(-\frac{qx^2}{2})\,dx +$$

$$+\int_{\frac{l}{2}}^{l} \frac{1}{EJ}\,1\,(-\frac{qx^2}{2})\,dx = \frac{-q}{2EJ}\,\frac{x^3}{3}\bigg|_{\frac{l}{2}}^{l} = \frac{-7ql^3}{48EJ}.$$

For the same example, when calculating the area of the deflection diagram using the auxiliary state 3 (the beam is loaded with a unit uniformly distributed load), we obtain:

$$M_F = -\frac{qx^2}{2}, \quad \bar{M}_3 = -\frac{x^2}{2},$$

$$\omega = \Delta_{3F} = \int_0^l \frac{1}{EJ}\left(-\frac{qx^2}{2}\right)\left(-\frac{x^2}{2}\right)dx = \frac{q}{4EJ}\,\frac{x^5}{5}\bigg|_0^l = \frac{ql^5}{20EJ}.$$

The indicated method of calculating the Mohr integrals can lead to significant difficulties, since a very complex formula either can be obtained, or cannot be obtained at all for an indefinite integral $F(x)$.

In practice, integrals, such as $\int_a^b \frac{f_1(x)\, f_2(x)}{f_3(x)}\, dx$ are calculated graph-analytically or using numerical integration.

But for the case when the bar has constant rigidity in the integration area, that is $EJ = f_3(x) = \text{const}$, and one of the function $f_1(x)$ or $f_2(x)$ is linear, the method proposed by A. K. Vereshchagin is usually used. This method is one of the most effective methods of calculating definite integrals. Let us explain its essence.

We plot the graphs of functions $f_1(x)$ and $f_2(x)$, that is, diagrams of bending moments $\bar{M}_i(x)$ and $M_F(x)$ on the integration area (Figure 7.18).

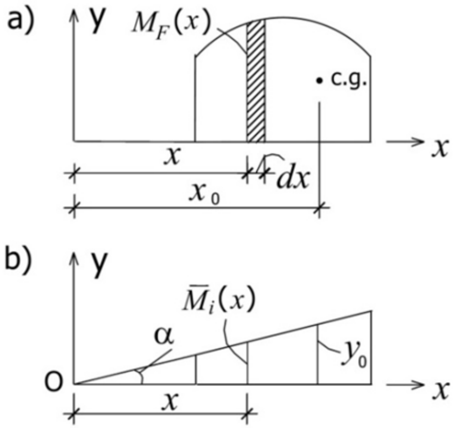


Figure 7.18

Suppose, for example, diagram $\bar{M}_i$ is rectilinear (Figure 7.18, b). The reference point is the intersection point of the bar axis with the diagram inclined line. Then $\bar{M}_i(x) = x\ \text{tg}\alpha$, and the Mohr integral is converted to

$$\int_a^b \frac{\bar{M}_i M_F \, dx}{EJ} = \frac{1}{EJ}\int_a^b x \, tg\alpha \cdot M_F \, dx = \frac{tg\alpha}{EJ}\int_a^b x \, M_F \, dx.$$

The integral $\int_a^b x \, M_F \, dx$, by definition, is the static moment of the area of the diagram M_F (Figure 7.18, a) relative to the axis y. The static moment is equal to the product of the area of this diagram by the distance from its center of gravity to the axis, that is:

$$\int_a^b x \, M_F \, dx = \omega \, x_0.$$

Given the ratio $x_0 = y_0/\text{tg}\alpha$, we get:

$$\int_a^b \frac{\bar{M}_i M_F \, dx}{EJ} = \frac{\omega \, y_0}{EJ}. \tag{7.9}$$

Thus, the Mohr integral is calculated by multiplying the area of the curvilinear diagram with the ordinate of the rectilinear diagram, taken under the center of gravity of the curvilinear one.

The process of calculating the integrals by Vereshchagin's method is sometimes called the "multiplication" of diagrams. The positive sign of the product ωy_0 is taken when the diagram M, whose area is denoted by ω, and the ordinate y have the same signs, i.e., when they are located on one side of the bar. In practice, one can be guided by a simpler rule: if both diagrams of efforts for certain section of the bar are located on one side of its axis, the result of their "multiplying" is accepted as positive, if diagrams are located on opposite sides of the bar, the result of their "multiplying" is accepted as negative.

When using the Vereshchagin's rule, complex diagrams of the internal forces should be represented as a sum of simple ones, for each of which formulas for area calculation and gravity center position are known. Examples of the simple diagrams are bending moment diagrams for cantilever or single-span beams loaded with concentrated force or uniformly distributed load (Figure 7.19).

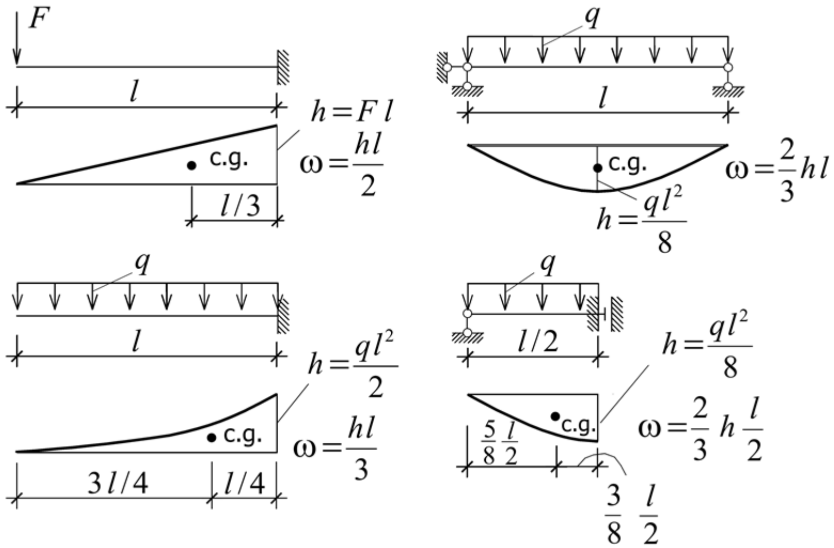


Figure 7.19

To obtain simple diagrams, the principle of independence of the action of forces should sometimes be used.

E x a m p l e. Determine the vertical displacements of points A and B (Figure 7.20) of the beam with constant rigidity.

Diagrams of bending moments for a beam from a given load and unit forces are shown in Figure 7.20.

$$\Delta_{1F} = \sum\int \frac{\bar{M}_1 M_F dx}{EJ} = \sum \frac{\omega\, y_0}{EJ} = \frac{1}{EJ}\frac{1}{2} Fa \cdot 2a \cdot a = \frac{Fa^3}{EJ}.$$

$$\Delta_{2F} = \sum\int \frac{\bar{M}_2 M_F dx}{EJ} = \sum \frac{\omega\, y_0}{EJ} = \frac{1}{EJ}\frac{1}{2} Fa \cdot a \cdot \frac{1}{3} a = \frac{Fa^3}{6EJ}.$$

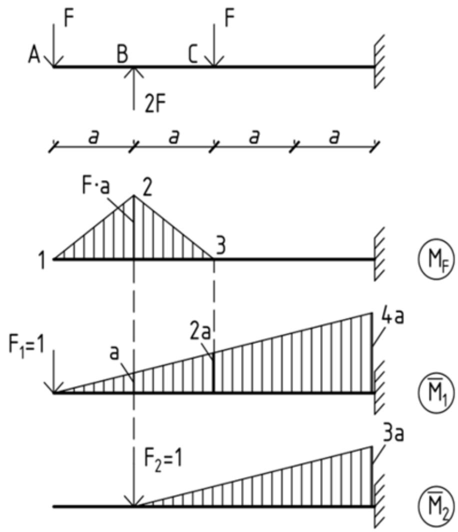


Figure 7.20

Ex a m p l e. Determine the vertical displacement of the point D and the angle of rotation of the cross section C of the beam with constant rigidity (Figure 7.21, a).

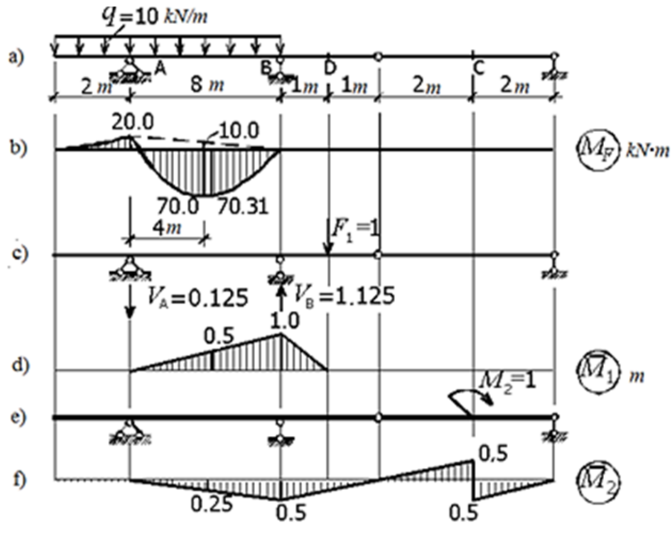


Figure 7.21

To determine the vertical displacement of a point D we load the beam with force $F_1 = 1$ (Figure 7.21, c) and construct the corresponding diagram of bending moments (Figure 7.21, d).

Using the principle of superposition, we represent the diagram M_F in the form of two simple ones (Figure 7.22) and determine the displacement according to the Vereshchagin's rule:

$$\Delta_{1F} = \sum \int \frac{\bar{M}_1 M_F dx}{EJ} = \sum \frac{\omega\, y_0}{EJ} =$$
$$= \frac{1}{EJ} \frac{1}{2} 20 \cdot 8 \frac{1}{3} - \frac{1}{EJ} \frac{2}{3} 80 \cdot 8 \cdot 0.5 = -\frac{186.67}{EJ} \ \text{m.}$$

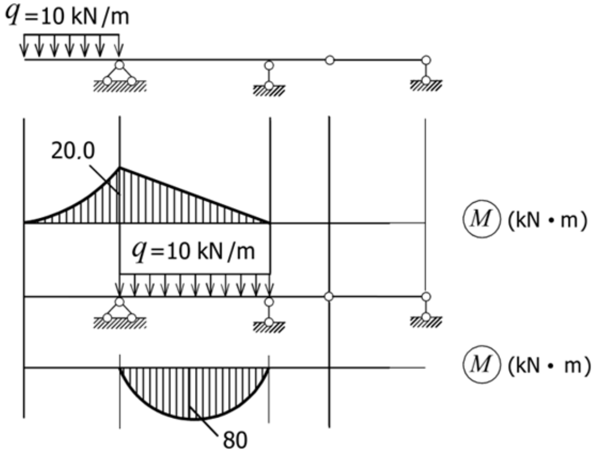


Figure 7.22

The auxiliary state for determining the angle of rotation of cross-section C is shown in Figure 7.21,e, and the corresponding diagram of bending moments is shown in Figure 7.21,f.

$$\Delta_{2F} = \sum \int \frac{\bar{M}_2 M_F dx}{EJ} = -\frac{1}{EJ} \frac{1}{2} 20 \cdot 8 \frac{1}{3} 0.5 +$$
$$+ \frac{1}{EJ} \frac{2}{3} 80 \cdot 8 \cdot 0.25 = \frac{93.33}{EJ} \ \text{rad.}$$

E x a m p l e. Find the horizontal displacement of point A of the frame shown in Figure 7.23 a.

The auxiliary state (state 1) is shown in Figure 7.22, b. The bending moment diagrams corresponding to the frame states are shown in Figure 7.23, c, d.

$$\Delta_{1F} = \sum\int \frac{\bar{M}_1 M_F \, dx}{EJ} = \frac{1}{EJ}\frac{1}{3}16\cdot 4\cdot 3 + \frac{1}{2EJ}6\cdot 4\frac{16+32}{2} +$$

$$+\frac{1}{EJ}\frac{1}{2}32\cdot 4\frac{2}{3}4 = \frac{1568}{3EJ} \text{ m.}$$

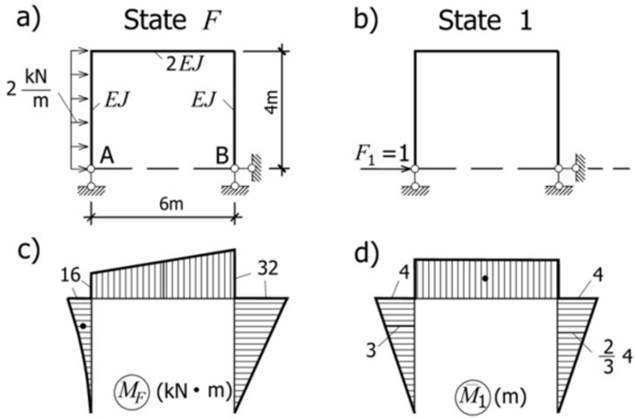


Figure 7.23

In this example, the required displacement is calculated as the sum of the integrals over three members. In each of them, the functions $\bar{M}_1(x)$ and $M_F(x)$ have well-defined analytical expressions. If through the length of one element the diagrams of moments are described by different functional dependencies, the element must be divided into the corresponding sections, the integrals must be calculated separately for each section, and the calculation results should be summarized.

Once again, we note that the Vereshchagin's method cannot be applied in the case when both diagrams are non-linear. So, for example, it cannot be applied to calculating the area of the diagram of the deflections of a beam loaded with a uniformly distributed load.

The same rule for calculating integrals can be applied to the other two terms in the Mohr's formula for determining displacements.

The value of a definite integral, as it is known, can be calculated using formulas of numerical integration, that are based on replacing the integral with a finite sum:

$$\int_a^b f(x)\,dx \approx \sum_{k=0}^{n} c_k f\left(x_k\right),$$

where x_k are the points of the segment $[a, b]$:

c_k are the numerical coefficients.

Given equality, generally approximate, is called the quadrature formula, points x_k are the nodes of the quadrature formula, and numbers c_k are called coefficients of the quadrature formula. The error of the quadrature formula

$$\psi = \int_a^b f(x)\,dx - \sum_{k=0}^{n} c_k f\left(x_k\right)$$

depends both on the location of the nodes and on the choice of coefficients. Most often, a uniform grid of nodes is used in practical applications to the problems of structural mechanics; in this case, the initial integral is represented as the sum of the integrals over partial segments, on each of which a quadrature formula is applied.

The simplest quadrature formulas for one interval are the rectangle formula

$$\int_a^b f(x)dx \approx (b-a)\cdot f\left(\frac{b+a}{2}\right)$$

and the trapezoid formula

$$\int_a^b f(x)dx \approx (b-a)\,\frac{f(a)+f(b)}{2}.$$

Naturally, even in the case of functions close to linear, the use of these formulas will lead to an error in the calculations of displacements.

If concentrated forces or uniformly distributed load act on a system composed of rectilinear elements, the diagram of bending moments on separate sections of the element is limited to a straight line or parabola. If it is necessary for this system to determine the linear or angular displacement of some point, in the auxiliary state, the contour of the diagram "M" due to the load $F_1 = 1$ will be determined by linear relationships $M(x)$. In this case, when $f_3(x) = \text{const}$, then the function $f(x) = f_1(x)\, f_2(x)$ will be represented by a curve of the second or third degree. Then, on the segments of elements with constant rigidity, the Mohr integral can be calculated exactly using T. Simpson's formula (parabola formula):

$$\int_a^b f(x)dx = \frac{l}{6}\left(y_1 + 4y_2 + y_3\right), \tag{7.10}$$

where y_1, y_2, y_3 are the values of the function at the end points of the segment and in the middle of it (Figure 7.24).

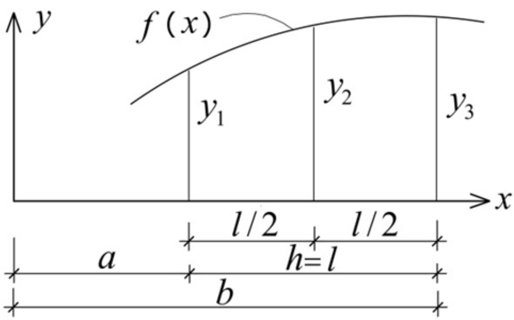


Figure 7.24

Simpson's formula is exact for any polynomial not higher than the third degree.

Using the Simpson's formula, we determine the vertical displacement of the cross-section D and the angle of rotation of the cross-section C for the beam shown in Figure 7.21:

$$\Delta_{1F} = \sum\int \frac{\bar{M}_1 M_F dx}{EJ} = \frac{8}{6EJ}(0 - 4\cdot 70\cdot 0.5 + 0) = -\frac{186.67}{EJ}, m;$$

$$\Delta_{2F} = \frac{8}{6EJ}(0 + 4\cdot 70\cdot 0.25 + 0) = \frac{93.33}{EJ}, rad.$$

The obtained values of the displacements coincide with those ones found according to the Vereshchagin's rule.

E x a m p l e. Determine the angle of mutual rotation of the ends of the beams, adjacent to the hinge C (Figure 7.25). The bending rigidity of the beams is constant.

Diagrams of bending moments for a beam from a given load and unit force are shown in Figure 7.25.

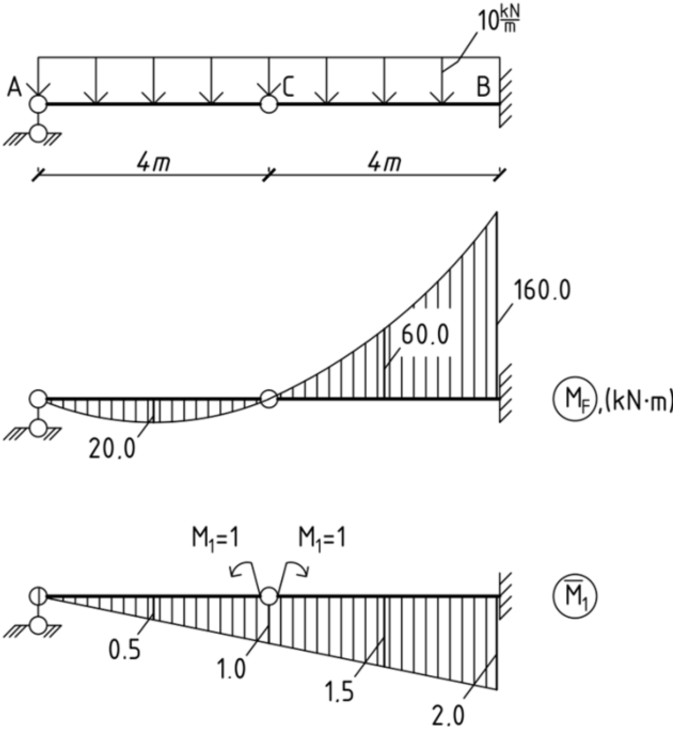


Figure 7.25

$$\Delta_{1F} = \sum\int \frac{\bar{M}_1 M_F dx}{EJ} = \frac{1}{EJ}\cdot\frac{2}{3}\cdot 20\cdot 4\cdot 0.5 +$$

$$+\frac{4}{6EJ}(-4\cdot 1.5\cdot 60 - 2\cdot 160) = -\frac{1280}{3EJ}\cdot$$

We offer the reader to show how the same value of the displacement can be calculated easier.

E x a m p l e. Determine the horizontal displacement of the end of the cantilever broken beam (Figure 7.26, a).

The diagram of bending moments caused by a given load is shown in Figure 7.26, b, from unit force $F_1 = 1$ is shown in Figure 7.26, c.

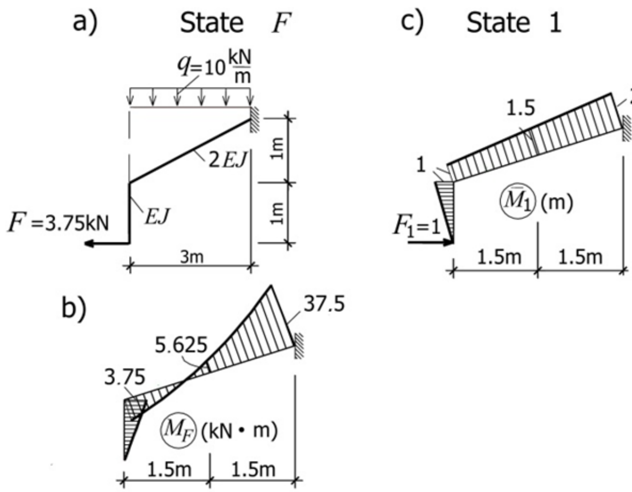


Figure 7.26

“Multiplication” of diagrams on a vertical element is made according to the Vereshchagin's rule, on an inclined one (its length is $\sqrt{10}$ m) – according to Simpson's formula:

$$\Delta_{1F} = -\frac{1}{EJ}\frac{1}{2}1\cdot 3.75\frac{2}{3} + \frac{\sqrt{10}}{6\cdot 2EJ}\left(-1\cdot 3.75 + 4\cdot 5.625\cdot 1.5 + 37.5\cdot 2\right) =$$

$$= \frac{-1.25 + 8.75\sqrt{10}}{EJ} = \frac{26.42}{EJ} \text{ m.}$$

If a function in a certain section of the element is a more complex than a polynomial of the third degree, which is possible for elements of curvilinear shape, or the rigidity changes along the axis of the element, or the load is non-uniformly distributed on it, the result of the calculation using the Simpson's formula will be approximate.

On a partial section, the error is estimated as follows:

$$|\psi| \le \frac{h^5}{2880} M,$$

where

$$M = \sup_{x \in [a,b]} \left| f^{IV}(x) \right|^{*},$$

that is, on this section the Simpson's formula has accuracy $O\left(h^5\right)$, on the whole section accuracy is $O\left(h^4\right)$, while the trapezoid formula, like the formula of rectangles, has a second order of accuracy.

Example. Using the Simpson's formula, determine the area of the deflection's diagram of the cantilever beam with a constant cross-section, loaded with a uniformly distributed load.

Diagrams M_F and $\overline{M}_1$ are shown in Figure 7.27.

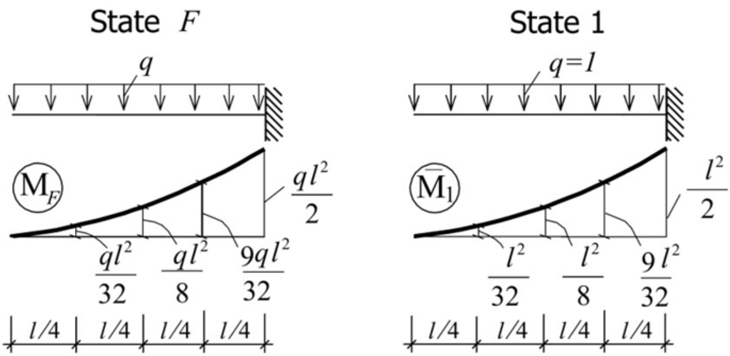


Figure 7.27

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* Lat. *supremus* is the highest.

Here:

$$f(x) = M_F(x)\,\bar{M}_1(x) = \frac{qx^4}{4}.$$

For the variant with one section of length l we get:

$$\Delta_{1F} = \frac{l}{6EJ}\left(4\frac{ql^2}{8}\frac{l^2}{8} + \frac{ql^2}{2}\frac{l^2}{2}\right) = \frac{ql^5}{19.2\,EJ}.$$

The exact solution has been obtained earlier by direct integration. Area is $\Delta_{1F} = \dfrac{ql^5}{20\,EJ}$.

If we accept $\dfrac{q}{EJ} = 1$, then the calculation error is $\psi = \dfrac{l^5}{19.2} - \dfrac{l^5}{20} =$ $= 2.083 \cdot 10^{-3}\, l^5$, which corresponds to the previously given estimation $\dfrac{l^5}{2880} M = \dfrac{l^5}{2880} 6 = 2.083 \cdot 10^{-3}\, l^5$, where it is accepted, that:

$$M = \sup_{x \in [a,b]} \left| f^{IV}(x) \right| = \sup_{x \in [a,b]} \left| \left(\frac{x^4}{4} \right)^{IV} \right| = 6.$$

For the variant with two sections of length $\dfrac{l}{2}$ we get:

$$\Delta_{1F} = \frac{l}{2 \cdot 6EJ}\left(4\frac{ql^2}{32}\frac{l^2}{32} + \frac{ql^2}{8}\frac{l^2}{8}\right) +$$

$$+\frac{l}{2 \cdot 6EJ}\left(\frac{ql^2}{8}\frac{l^2}{8} + 4\frac{9ql^2}{32}\frac{9l^2}{32} + \frac{ql^2}{2}\frac{l^2}{2}\right) \approx \frac{ql^5}{19.95\,EJ}.$$

The error is equal $\psi = \dfrac{l^5}{19.95} - \dfrac{l^5}{20} = 1.253 \cdot 10^{-4}\, l^5$. On the entire integration interval, the error is estimated as follows:

$$|\psi| \le \frac{h^4(b-a)}{2880} M.$$

In this case $h = \frac{l}{2}$, $b - a = l$ and, therefore, $\psi < \frac{l^5}{16 \cdot 2880} 6 =$ $= 1.302 \cdot 10^{-4}\, l^5$.

The Simpson's formula is set on three equally spaced nodes. In some cases, quadrature formulas are applied with a large number of equally spaced nodes. In particular, such a formula, built on four nodes, is the following one:

$$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a) + 3f\left(a + \frac{b-a}{3}\right) + 3f\left(a + \frac{2(b-a)}{3}\right) + f(b)\right].$$

This formula is sometimes convenient to use to multiply linear diagrams of the internal forces. The result of this calculation is accurate. For example, if the multiplied diagrams have the form shown in Figure 7.28, the Mohr's integral in this section will be equal to:

$$\frac{1}{EJ}\int_0^l f_1(x)\, f_2(x)dx = \frac{l}{8EJ}\left(-a\, c - b\, d\right).$$

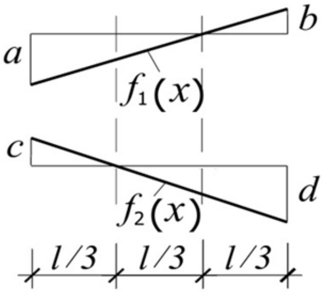


Figure 7.28

In general, formulas with a large number of equally spaced nodes are applied relatively rarely.

7.8. Determining Displacements Caused by the Thermal Effects

Suppose that for a system in state *a* (Section 7.6) the external influence is thermal one: the temperature of systems elements has changed with respect to some initial state. For an infinitely small element (Figure 7.29) of this system, we take the temperature of the lower fiber equal to t_1, the upper one equal to t_2. And the temperature distribution along the cross-section height is accords to the linear law.

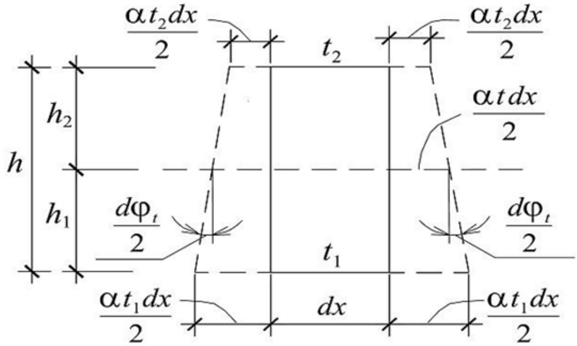


Figure 7.29

The temperature on the axis passing through the center of gravity of the cross-section will be equal $t = t_2 + \frac{t_1 - t_2}{h} h_2$. When $h_1 = h_2$ we get $t = \frac{t_1 + t_2}{2}$.

Under the influence of temperature, the element moves to a new position (it is indicated by a dashed line). In this new position, all the fibers are extended by an amount $d\varepsilon_t = \varepsilon\, dx = \alpha t\, dx$ and each lateral face is rotated by an angle $\frac{d\varphi_t}{2}$ relative to the axis passing through the center of gravity.

The elongation of the lower fiber is equal to $\alpha t_1\, dx$, and the upper one is equal to $\alpha t_2\, dx$, (α is the coefficient of linear expansion). Then, due to small deformations, we obtain:

$$d\varphi_t = k\,dx = \frac{\alpha t_1\,dx - \alpha t_2\,dx}{h} = \frac{\alpha(t_1 - t_2)}{h}dx = \frac{\alpha t' dx}{h},$$

where $t' = t_1 - t_2$ is the temperature difference.

Since temperature deformations do not cause a cross-sectional shear, substituting $d\varepsilon_t$ and $d\varphi_t$ in the general formula (7.6) for determining displacements and replacing the index a in the designation Δ_{ia} by t (indicates the reason that caused the displacement), we obtain:

$$\Delta_{it} = \sum\int_l \bar{N}_i\,\alpha t\,dx + \sum\int_l \bar{M}_i \frac{\alpha t'}{h}dx. \tag{7.11}$$

Note that each of the integrals in this expression determines the work of the internal forces of the auxiliary state of the system on displacements caused by a change in temperature. Therefore, the values of the integrals are accepted positive on the integration interval in the case when the corresponding directions of the element deformations, caused by the forces of the i-th (auxiliary) state and by thermal action, coincide.

If the values α, t, t' and h remain unchanged in some parts of the elements, the expression (7.11) is converted to the form:

$$\Delta_{it} = \sum\alpha\,t\,\Omega_N + \sum\frac{\alpha\,t'}{h}\,\Omega_M, \tag{7.12}$$

where

$$\Omega_N = \int_l \bar{N}_i\,dx, \quad \Omega_M = \int_l \bar{M}_i\,dx$$

are the areas of the diagrams of longitudinal forces and bending moments on the segments of the members with the specified features.

E x a m p l e. Determine the horizontal displacement of the frame support B (Figure 7.30, a) from thermal action indicated on the figure. Unchanged cross-sections through the length of each element are assumed to be symmetrical. The height of the vertical element is h_1, the height of the horizontal one is h_2.

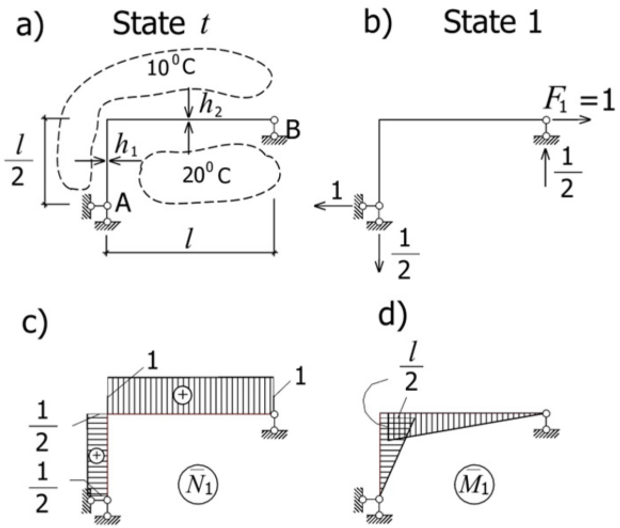


Figure 7.30

The temperature along the axis of each member is $t = \dfrac{20+10}{2} = 15^{\circ}$, the temperature difference is $t' = 20 - 10 = 10^{\circ}$.

The auxiliary state of the frame is shown in Figure 7.30, b, and the diagrams of internal forces $\bar{N}_1$ и $\bar{M}_1$ are shown in Figure 7.30, c, d.

We calculate the required displacement:

$$\Delta_{1t} = \sum \alpha t \Omega_{N_1} + \sum \frac{\alpha t'}{h} \Omega_{M_1} = \alpha\, 15 \frac{1}{2}\frac{l}{2} + \alpha\, 15 \cdot 1 \cdot l +$$

$$+ \frac{\alpha\, 10}{h_1}\frac{1}{2}\frac{l}{2}\frac{l}{2} + \frac{\alpha\, 10}{h_2}\frac{1}{2}\frac{l}{2} l = 18.75\alpha + \left(\frac{1.25 l^2}{h_1} + \frac{2.5 l^2}{h_2} \right) \alpha.$$

E x a m p l e. Determine the angular displacement of the frame cross-section K (Figure 7.31, a) from the thermal action indicated on the figure. Unchanged cross-sections through the length of each element are assumed to be symmetrical. The height of vertical and horizontal elements is $h = 0.6\, m$. The coefficient of linear expansion is $\alpha = 10 \cdot 10^{-6} \left({}^{\circ}C^{-1} \right)$.

The temperature along the axis of members is:

$$t_{AB} = t_{BD} = \frac{-10+20}{2} = 5^{\mathrm{o}}; \quad t_{CD} = \frac{0+20}{2} = 10^{\mathrm{o}};$$

$$t_{DE} = \frac{-10+0}{2} = -5^{\mathrm{o}}; \quad t_{CK} = \frac{0+0}{2} = 0^{\mathrm{o}}.$$

the temperature differences are:

$$t'_{AB} = t'_{BD} = 20-(-10) = 30^{\mathrm{o}}; \quad t'_{CD} = 20-0 = 20^{\mathrm{o}};$$

$$t'_{DE} = 0-(-10) = 10^{\mathrm{o}}; \quad t'_{CK} = 0-0 = 0^{\mathrm{o}}.$$

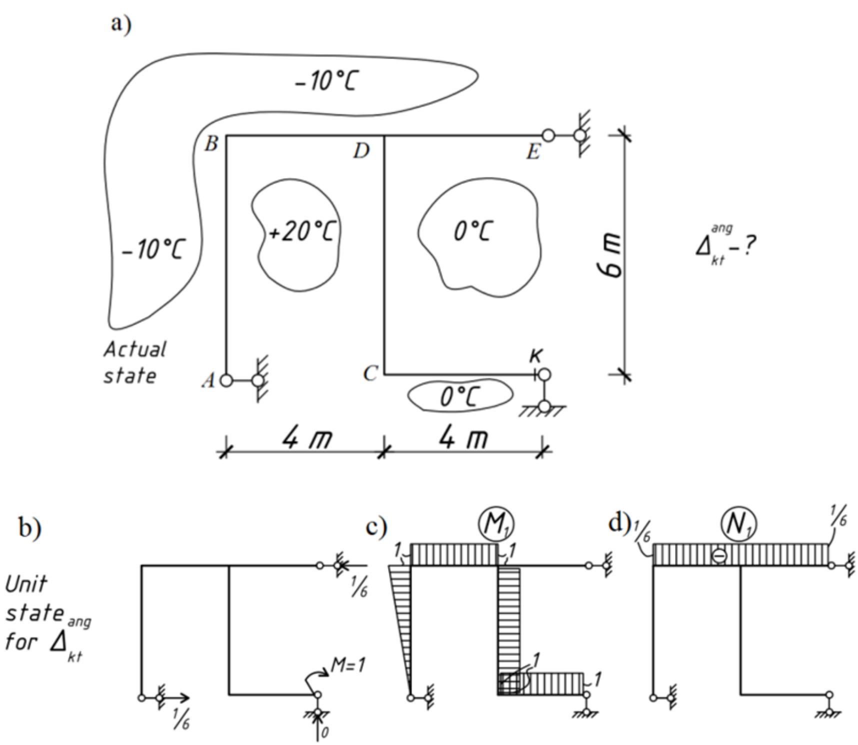


Figure 7.31

The auxiliary state of the frame and the diagrams of internal forces M_1 and N_1 are shown in Figure 7.31, b, c, d.

We calculate the required displacement:

$$\Delta_{1t} = \sum \alpha t \Omega_{N_1} + \sum \frac{\alpha t'}{h} \Omega_{M_1} = 10 \cdot 10^{-6} \left(5 \cdot 0 + 5 \cdot \left(-\frac{1}{6} \cdot 4 \right) + 10 \cdot 0 + \right.$$

$$+(-5) \cdot \left(-\frac{1}{6} \cdot 4 \right) + 0 \cdot 0 \cdot l + \left(-\frac{30}{0.6} \cdot \frac{1}{2} \cdot 1 \cdot 6 \right) + \left(-\frac{30}{0.6} \cdot 1 \cdot 4 \right) +$$

$$\left. + \left(-\frac{20}{0.6} \cdot 1 \cdot 6 \right) + \frac{10}{0.6} \cdot 0 + \frac{0}{0.6} \cdot 1 \cdot 4 \right) = -0.0055\, rad.$$

7.9. Determination of Displacements Caused by the Settlement of Supports

Suppose that the support connections of a given statically determinate system (Figure 7.32, a) under the influence of some actions moves to the positions shown in Figure 7.32, a: rigid support turned clockwise by an angle c_1, and the hinged-movable support moved upward by c_2. We denote this state of the system as state c. To determine the displacement of a point, for example, the horizontal displacement of the node D, we apply a force $F_i = 1$ in the auxiliary state in the direction of the required displacement (Figure 7.32, b).

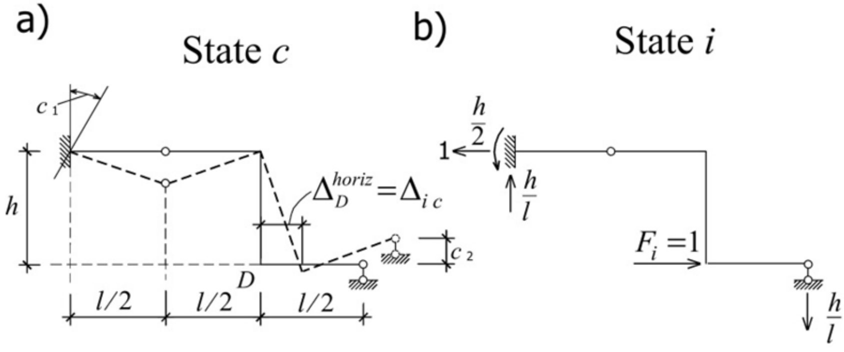


Figure 7.32

We define the work of the forces of the i-th state of the system at its displacements in the state c. There are no internal forces in a state c: displacements of the supports of a statically determinate system do not cause forces in its elements. Therefore, only external forces, which include support reactions, will do the work on the displacements of the state c. In accordance with the principle of virtual displacements, we obtain:

$$1 \cdot \Delta_{ic} + \sum R_{ki}\, c_k = 0,$$

where R_{ki} is the reaction in k-th support link caused by $F_i = 1$;

c_k is the given displacement of link k.

So it follows that

$$\Delta_{ic} = -\sum R_{ki}\, c_k. \tag{7.13}$$

The sign of the product $R_{ki}\, c_k$ is assumed to be positive if the directions of R_{ki} and c_k coincide.

For this example, we get:

$$\Delta_D^{horiz} = \Delta_{ic} = -\sum R_{ki}\, c_k = -\left(-\frac{h}{2} c_1 - \frac{h}{l} c_2\right) = h\left(\frac{c_1}{2} + \frac{c_2}{l}\right).$$

E x a m p l e. Determine the horizontal displacement of the frame cross-section K (Figure 7.33, a) caused by the settlement of supports indicated on the figure.

According to (7.13), the expression for the requied displacement is:

$$\Delta_{KC}^{horiz} = -\sum R_{ki}\, c_k.$$

The auxiliary state of the frame for determining support reactions caused by a unit concentrated force applied to the cross-section K in the horizontal direction is shown in Figure 7.33, b.

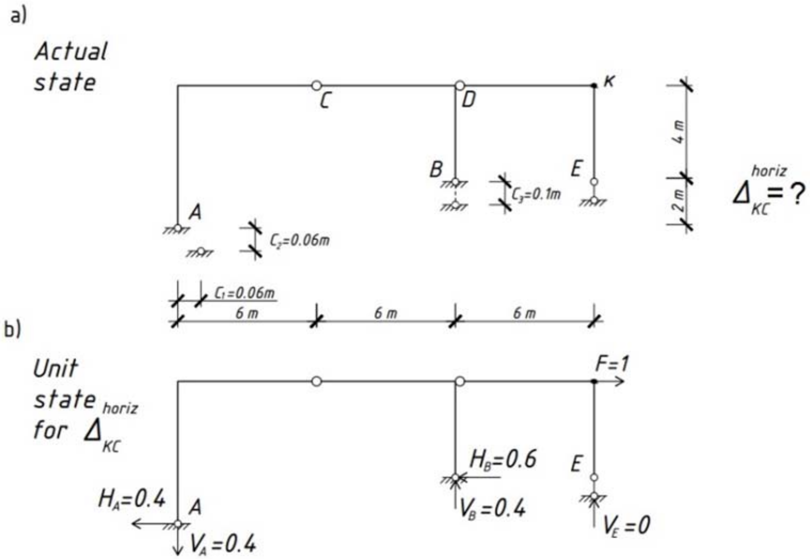


Figure 7.33

A given system is a statically determinate compound frame. We find support reactions from equilibrium equations:

$$\sum M_D = 0: \quad V_E \cdot 6 = 0 \quad \Rightarrow \quad V_E = 0\,kN;$$

$$\left.\begin{array}{l} \sum M_A = 0: \quad -V_B \cdot 12 - H_B \cdot 2 + 1 \cdot 6 = 0 \\ \sum M_C^{right} = 0: \quad -V_B \cdot 6 + H_B \cdot 4 = 0 \end{array}\right\} \Rightarrow V_B = 0.4\,kN, H_B = 0.6\,kN;$$

$$\left.\begin{array}{l} \sum M_B = 0: \quad -V_A \cdot 12 + H_A \cdot 2 + 1 \cdot 4 = 0 \\ \sum M_C^{left} = 0: \quad -V_A \cdot 6 + H_A \cdot 6 = 0 \end{array}\right\} \Rightarrow V_A = 0.4\,kN, H_A = 0.4\,kN.$$

We calculate the required displacement:

$$\Delta_{KC}^{horiz} = -\left(0.4 \cdot 0.06 - 0.4 \cdot 0.06 - 0.4 \cdot 0.1 + 0.6 \cdot 0 + 0 \cdot 0\right) = 0.04\,m.$$

Here the «–» sign is accepted before $H_A\,c_1$ and $V_B\,c_3$, since the direction of the reaction and the corresponding settlement do not coincide.

In conclusion, we note that if a given linearly deformable system is simultaneously exposed to external load, temperature changes, the displacement of supports or other exposures, the required total displacement is determined by summing the components from each exposure separately.

The features of determining displacements in statically indeterminate systems will be described below.

7.10. Matrix Form of the Displacements Determination

Consider this question in relation to the plane trusses. In practical problems of trusses calculating, it is important to be able to determine the displacements of each node in horizontal and vertical directions. The total number of unknown displacements with this approach will be equal to the number of degrees of freedom of the nodes $m = 2N - L$ (there are no displacements of nodes in the directions of the support links). In Figure 7.34, a unknown displacements of nodes are shown by arrows.

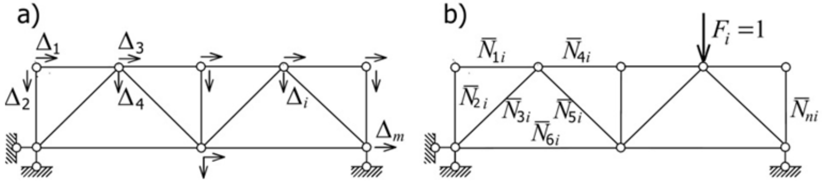


Figure 7.34

To determine the displacement Δ_i we take the auxiliary state as shown in Figure 7.34, b: load $F_i = 1$ is applied in the direction of the required displacement. In this figure, a designation of the force $\overline{N}_{ki}$ arising in the rods is shown near each rod of the truss, where the index k corresponds to the number of the rod. The index n corresponds to the number of the last truss member.

From formula (7.6) it follows that

$$\Delta_i = \sum \overline{N}_i \int_0^l \varepsilon\, dx = \sum_{k=1}^{n} \overline{N}_{ki}\, \Delta l_k ,$$

where $\overline{N}_{ki}$ is the force in the k -th rod caused by $F_i = 1$;

Δl_k is absolute deformation of the k -th truss rod.

An expanded record of the last expression with respect to all calculated displacements will appear as the following equations:

$$\Delta_1 = \overline{N}_{11}\,\Delta l_1 + \overline{N}_{21}\,\Delta l_2 + \ldots + \overline{N}_{n1}\,\Delta l_n,$$
$$\Delta_2 = \overline{N}_{12}\,\Delta l_1 + \overline{N}_{22}\,\Delta l_2 + \ldots + \overline{N}_{n2}\,\Delta l_n,$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$\Delta_m = \overline{N}_{1m}\,\Delta l_1 + \overline{N}_{2m}\,\Delta l_2 + \ldots + \overline{N}_{nm}\,\Delta l_n$$

or in matrix form:

$$\vec{\Delta} = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_m \end{bmatrix} = \begin{bmatrix} \overline{N}_{11} & \overline{N}_{21} & \ldots & \overline{N}_{n1} \\ \overline{N}_{12} & \overline{N}_{22} & \ldots & \overline{N}_{n2} \\ \vdots & \vdots & \ldots & \vdots \\ \overline{N}_{1m} & \overline{N}_{2m} & \ldots & \overline{N}_{nm} \end{bmatrix} \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \\ \vdots \\ \Delta l_n \end{bmatrix} = L_N^T \vec{\Delta} l, \qquad (7.14)$$

where $\vec{\Delta}$ is the vector of nodal displacements;

L_N^T is the matrix transposed with respect to the influence matrix L_N;

$\vec{\Delta} l$ is the vector of absolute deformations of the rods.

For statically determinate truss $m = 2N - L = B,$ that is $m = n$ and in this case the matrix L_N will be square.

So, in order to find the displacements of the truss nodes, it is necessary to know the deformations Δl of the rods, determined in accordance with the action set on the system.

When the temperature changes:

$$\Delta l_k = \alpha\, t_k\, l_k,$$

where α is the coefficient of linear thermal expansion;

t_k is the temperature change of the k-th rod.

If there are displacements due to inaccuracy in the manufacture of the rods, Δl_k is determined as the differences between the real and design values of the lengths of the rods.

When calculating a physically nonlinear system under the action of a load F, it is possible, using a nonlinear tensile (compression) diagram, to determine the corresponding elongation (shortening) Δl_k by a known effort N_{kF}.

If the material of the rods at a given load F works in a linearly elastic stage, then:

$$\Delta l_k = \frac{N_{kF}\, l_k}{EA_k} = d_k\, N_{kF},$$

where EA_k is the rigidity of the rod in tension (compression);

$d_k = \dfrac{l_k}{EA_k}$ is the coefficient of pliability of the *k*-th rod.

Then for the vector of deformations caused by a given load, there is a dependence:

$$\vec{\Delta l} = \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \\ \vdots \\ \Delta l_n \end{bmatrix} = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} N_{1F} \\ N_{2F} \\ \vdots \\ N_{nF} \end{bmatrix} = D\, \vec{N}_F, \tag{7.15}$$

where D is the matrix of internal pliability of truss rods;

$\vec{N}_F$ is the vector of efforts in the truss rods from the load F.

Substituting expression (7.15) into formula (7.14), we obtain a matrix notation of the formula for determining the nodal displacements of the truss due to the load F:

$$\vec{\Delta} = L_N^T\, D\, \vec{N}_F. \tag{7.16}$$

To determine the displacements of bended systems due to the load F, we will use the Simpson's formula. At the *k*-th section of the bar with variable bending rigidity, the Mohr's integral is written in the form:

$$\int_0^{l_k} \frac{\bar{M}_i M_F dx}{EJ} = \frac{l_k}{6}\left(\frac{\bar{M}_i^B M_F^B}{EJ^B} + 4\frac{\bar{M}_i^M M_F^M}{EJ^M} + \frac{\bar{M}_i^E M_F^E}{EJ^E} \right),$$

where the superscripts B, M and E indicate the values $\bar{M}_i$, M_F, ... and EJ at the beginning, middle and end of the integration section.

We represent this expression in matrix form:

$$\int_0^{l_k} \frac{\bar{M}_i M_F\,dx}{EJ} = \begin{bmatrix} \bar{M}_i^B & \bar{M}_i^M & \bar{M}_i^E \end{bmatrix} \begin{bmatrix} \frac{l_k}{6EJ^B} & & \\ & \frac{4l_k}{6EJ^M} & \\ & & \frac{l_k}{6EJ^E} \end{bmatrix} \begin{bmatrix} M_F^B \\ M_F^M \\ M_F^E \end{bmatrix} =$$

$$= L_{ki}^T D_k \vec{M}_{kF},$$

where D_k is the diagonal matrix of pliability for the k-th section.

For the variant of linear diagrams $\bar{M}_\mathrm{i}, M_F, \ldots$ we obtain:

$$\bar{M}_i^M = \frac{\bar{M}_i^B + \bar{M}_i^E}{2}, \quad M_F^M = \frac{M_F^B + M_F^E}{2},$$

and then, at $EJ =$ const, the computations in the section are reduced to:

$$\int_0^{l_k} \frac{\bar{M}_i M_F\,dx}{EJ} = \begin{bmatrix} \bar{M}_i^B & \bar{M}_i^E \end{bmatrix} \begin{bmatrix} \frac{2l_k}{6EJ} & \frac{l_k}{6EJ} \\ \frac{l_k}{6EJ} & \frac{2l_k}{6EJ} \end{bmatrix} \begin{bmatrix} M_F^B \\ M_F^E \end{bmatrix}.$$

Summing up the results of calculations for all sections, we obtain:

$$\Delta_{iF} = \sum \int \frac{\bar{M}_i M_F\,dx}{EJ} = \sum_k L_{ki}^T D_k \vec{M}_{kF}. \qquad (7.17)$$

Using the sequential docking of the bending moment vectors in all n parts of the system and introducing the matrix of pliability D for the entire system into the calculation, the displacements calculation can be represented as follows:

$$\Delta_{iF} = \begin{bmatrix} L_{1i}^T & L_{2i}^T \ldots L_{ni}^T \end{bmatrix} \times$$

$$\times \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{bmatrix} \begin{bmatrix} \vec{M}_{1F} \\ \vec{M}_{2F} \\ \vdots \\ \vec{M}_{nF} \end{bmatrix} = L_i^T D \vec{M}_F . \tag{7.18}$$

If it is necessary to determine the displacements of several points of the system, the row-vector L_i^T should be replaced by a matrix L^T, in each row of which values of bending moments caused by the *i*-th auxiliary state are recorded.

If the problem is to determine the displacements caused by different loadings, it is necessary to replace the vector $\vec{M}_F$ with a matrix, in each column of which values of efforts corresponded to a certain load are recorded.

With these remarks, the expression for determining the displacements of a bended system in the general case can be written as:

$$\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \ldots & \Delta_{1t} \\ \Delta_{21} & \Delta_{22} & \ldots & \Delta_{2t} \\ \vdots & \vdots & \ldots & \vdots \\ \Delta_{m1} & \Delta_{m2} & \ldots & \Delta_{mt} \end{bmatrix} =$$

$$= \begin{bmatrix} L_{11}^T & L_{21}^T & \ldots & L_{n1}^T \\ L_{12}^T & L_{22}^T & \ldots & L_{n2}^T \\ \vdots & \vdots & \ldots & \vdots \\ L_{1m}^T & L_{2m}^T & \ldots & L_{nm}^T \end{bmatrix} \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{bmatrix} \begin{bmatrix} \vec{M}_{1F}^{(1)} & \vec{M}_{1F}^{(2)} & \ldots & \vec{M}_{1F}^{(t)} \\ \vec{M}_{2F}^{(1)} & \vec{M}_{2F}^{(1)} & \ldots & \vec{M}_{2F}^{(t)} \\ \vdots & \vdots & \ldots & \vdots \\ \vec{M}_{nF}^{(1)} & \vec{M}_{nF}^{(1)} & \ldots & \vec{M}_{nF}^{(t)} \end{bmatrix} = \tag{7.19}$$

$$= L^T DM .$$

In this expression, the index m corresponds to the number of determined displacements for one loading, the index t corresponds to the number of independent loadings.

If $M = L$, the matrix Δ will be a matrix of external pliability A of the flexible bars system:

$$A = L_M^T \, D \, L_M . \qquad (7.20)$$

The same remark applies to formula (7.16). Replacing the vector $\vec{N}_F$ with the matrix $N = L_N$, as a result of the calculations we obtain the truss pliability matrix:

$$A = L_N^T \, D \, L_N . \qquad (7.21)$$

7.11. Influence Lines for Displacements

The theorem of reciprocal displacements is used to solve various problems in mechanics. In particular, the influence lines for displacements are relatively easy to obtain. Suppose, for example, it is necessary to construct the influence line for the rotation angle φ_k (Figure 7.35, a). Each new position of the unit force (Figure 7.35, b) corresponds to a certain value of the rotation angle ($\delta_{k1}, \delta_{k2}, ...$). At the same time, on the basis of the reciprocity theorem, these displacements can be determined each time by uploading the beam with a fixed generalized force $M_k = 1$ (Figure 7.35, c). Consequently, the shape of the influence lines for φ_k coincides with the diagram of the vertical displacements of the beam axis caused by force $M_k = 1$. The equation corresponding to this load for the bent axis of the beam is written in Section 7.5.

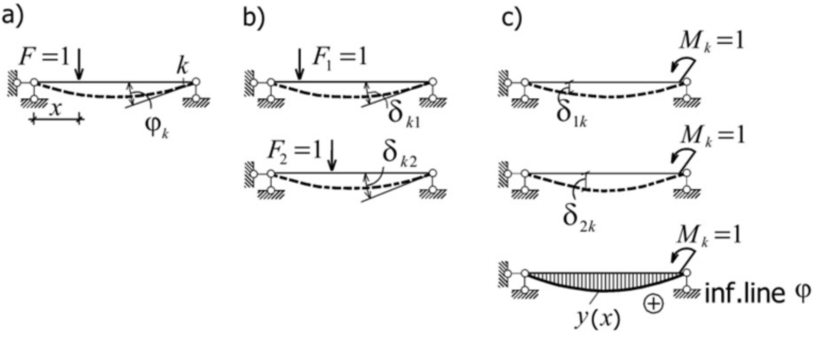


Figure 7.35

An analysis of the results of the last example (Figure 7.35) shows that the practical task of constructing influence lines for displacements of a linearly deformable system, on the one hand, can be associated with its calculation on the set of unit loads in characteristic sections, and then with the determination of the required displacement for each of them. On the other hand, this task may be connected with the calculation of the system for one load and the determination of the corresponding displacements in those cross sections in which the unknown shape of the influence line can be represented by the found displacements. The second solution is generally preferred.

We illustrate it with the example of a multi-span statically determinate beam (Figure 7.36), for which we will construct the influence line for δ_3. From the calculation of the loading beam by force $F_1 = 1$ we can find only one ordinate δ_{31} of the influence line for δ_3 (Figure 7.36, b), from the calculation at the action of the force $F_2 = 1$ we can find the ordinate δ_{32} and so on. A simpler technique is to construct an influence line δ_3 as a diagram of vertical displacements of the axis of the beam from the action of the force $F_3 = 1$ (Figure 7.36, c). In Figure 7.36, d it is shown the view of Inf. line for δ_3 taking into account generally accepted construction rules: positive ordinates are located above the axis of the beam, negative ones are below.

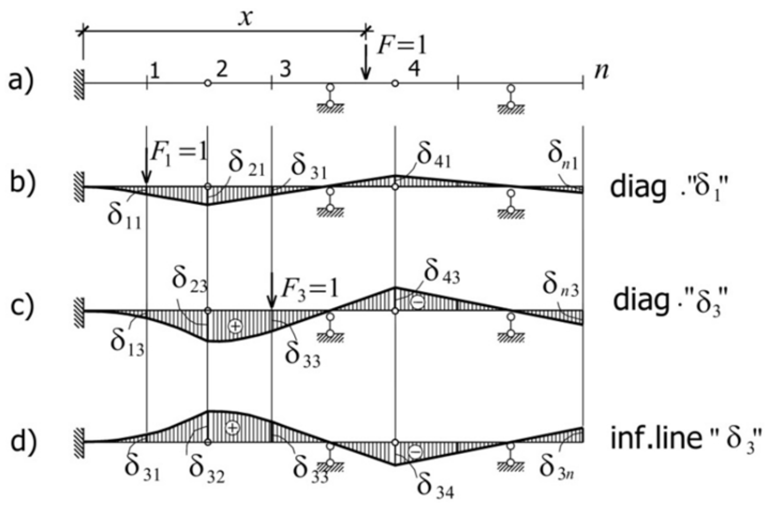


Figure 7.36

7.12. Influence Matrix for Displacements

The vertical displacement, due to the given load, of the cross-section i, for which the influence line for displacement is constructed, can be calculated by the formula:

$$\Delta_{iF} = \delta_{i1} F_1 + \delta_{i2} F_2 + \ldots + \delta_{in} F_n,$$

where $F_1, \ F_2, ..., \ F_n$ – are concentrated vertical forces applied in characteristic sections.

With the value of the index $i = 3$ we get the expression for calculation Δ_{3F} using the influence line (Figure 7.36, d).

Applying the expression for Δ_{iF} to each characteristic cross-section and using the matrix form for recording the transformations, we obtain the value of the displacement vector $\vec{\Delta}_F$:

$$\vec{\Delta}_F = \begin{bmatrix} \Delta_{1F} \\ \Delta_{2F} \\ \vdots \\ \Delta_{nF} \end{bmatrix} = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} & \ldots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \delta_{23} & \ldots & \delta_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \delta_{n1} & \delta_{n2} & \delta_{n3} & \cdots & \delta_{nn} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} = A\,\vec{F},$$

where
$$A = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} & \ldots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \delta_{23} & \ldots & \delta_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \delta_{n1} & \delta_{n2} & \delta_{n3} & \cdots & \delta_{nn} \end{bmatrix}$$

is the influence matrix for displacements.

The components of the *k*-th column are the ordinate values of the displacements diagrams constructed due to $F_k = 1$, which corresponds to the general definition of the influence matrices. Since the conditions $\delta_{ik} = \delta_{ki}$ are fulfilled, the matrix A is a symmetric matrix and, therefore the influence lines for δ_i can be constructed from the elements of the *i*-th column or *i*-th row.

In the case of systems of arbitrary outline, not necessarily the beams, displacements δ_{ik} may have different orientations in space. They determine the pliability of the system at some point i in a given direction (i-th) caused by the unit force applied at a point κ. Therefore, the matrix A is called the pliability matrix of the system. To calculate it, one can use formulas (7.20) and (7.21).

E x a m p l e . Calculate the matrix A of the external pliability of the frame in the given directions (Figure 7.37).

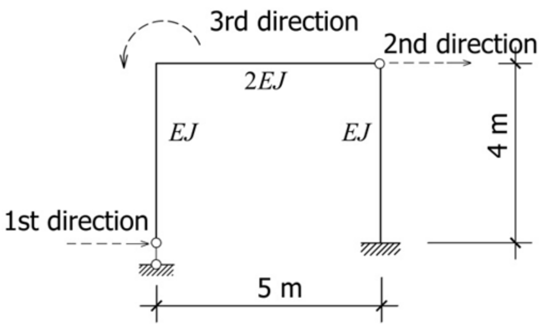


Figure 7.37

Diagrams of bending moments caused by the action of unit forces in given directions are shown in Figure 7.38.

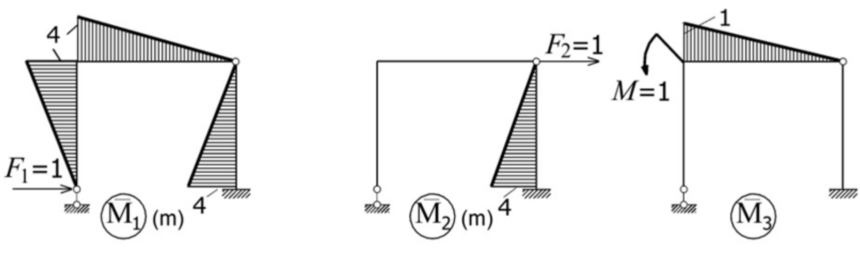


Figure 7.38

When compiling the influence matrix L_M, we will consider the ordinates of the diagrams M, located inside the frame contour as positive.

The pliability matrix A is calculated as follows:

$$A = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = L_M^T \; D \; L_M =$$

$$= \left|\begin{array}{ccc|ccc|ccc} 0 & -2 & -4 & -4 & -2 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & -1 & -0.5 & 0 & 0 & 0 & 0 \end{array}\right| \times$$

$$\times \left|\begin{array}{ccc|ccc|ccc} \frac{4}{6EJ} & & & & & & & & \\ & \frac{4\cdot 4}{6EJ} & & & & & & & \\ & & \frac{4}{6EJ} & & & & & & \\ \hline & & & \frac{5}{6\cdot 2EJ} & & & & & \\ & & & & \frac{4\cdot 5}{6\cdot 2EJ} & & & & \\ & & & & & \frac{5}{6EJ} & & & \\ \hline & & & & & & \frac{4}{6EJ} & & \\ & & & & & & & \frac{4\cdot 4}{6EJ} & \\ & & & & & & & & \frac{4}{6EJ} \end{array}\right| \times$$

$$\times \left|\begin{array}{ccc} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -4 & 0 & 0 \\ \hline -4 & 0 & -1 \\ -2 & 0 & -0.5 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right| = \left|\begin{array}{c|c|c} 168 & 64 & 10 \\ \hline 64 & 64 & 0 \\ \hline 10 & 0 & 2.5 \end{array}\right| \frac{1}{3EJ}.$$

THEME 8. FORCE METHOD AND ITS APPLICATION TO PLANE FRAMES CALCULATION

8.1. Statically Indeterminate Systems and Their Properties

Statically indeterminate systems are those systems in which not all internal forces can be found from the equilibrium equations.

In statically indeterminate systems, the number of unknown efforts exceeds the number of independent equilibrium equations. For example, to determine the four support reactions of the beam (Figure 8.1, a) arising from the action of any load on it, only three independent equilibrium equations can be compiled.

Consequently, in all cross-sections of the beam in the AC region, the internal forces cannot be determined. If in this beam we remove the support rod at a point B (Figures. 8.1, b) or introduce a hinge in a region BC (Figures. 8.1, c), then we obtain the design schemes of statically determinate beams. The constraints that can be removed from the beam (and in the general case, from any system) without changing its properties of geometrical unchangeability and unmoveability are called redundant constraints. The number of redundant constraints, the elimination of which turns the system into to the statically determinate one, is called the degree of static indeterminacy of the system (degree of redundancy). The beam shown in Figure 8.1, a, has statical indeterminacy of the first degree.

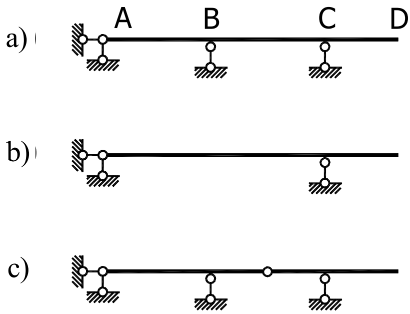


Figure 8.1

The same can be said about the design scheme of the truss (Figure 8.2). It is possible to find support reactions and forces in rods 3–5 and 4–5 caused by the load applied to its nodes solely using equilibrium equations, but the efforts of the rest of the rods remain unknown. Among these rods, there is one redundant, so the truss is statically indeterminate once.

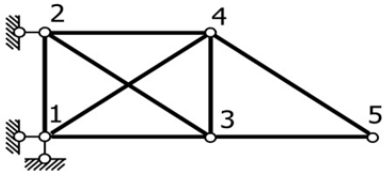


Figure 8.2

We note once again that the term “redundant constraint” should be understood from the point of view of the geometrical unchangeability and unmoveability of the system. According to the working conditions of the structure, these constraints are necessary; in their absence, the strength and rigidity of the structure may be insufficient.

Any constraint can be accepted as a redundant constraint, the elimination of which will not change the immutability and immobility of the system. So, for the scheme in Figure 8.1 as redundant constraint, you can take any vertical support rod or, in any cross-section on the region AC, the constraint, through which the bending moment is transmitted from one section of the beam to another.

The degree of the static indeterminacy of a structure is an important characteristic of a structure.

Statically indeterminate systems have the following properties.

1. The thermal effect on the system, the displacement of the supports or the inaccuracy of the manufacture of its elements with their subsequent tension during assembly cause, in the general case, additional forces in a statically indeterminate system. In a statically determinate system, these factors cause only displacements of the elements, while internal forces do not arise.

Here are some examples.

Let’s consider the temperature of the lower fibers of the beam (Figure 8.3, a) is equal t_1, and the upper ones is t_2, and $t_1 > t_2$. If there was no support link at the point B, then the cantilever beam AB due to the indicated action would have taken a position shown by a dashed line. To return the beam from this position to the initial (undeformed) position, it is necessary to apply a force X_1 at the point B', equals to the reaction that arises in the support B from temperature changes.

The displacement of the support point C to the position C' provokes bending of the beam AC (Figure 8.3, b), which indicates the appearance of bending moments and transverse forces in the beam cross-sections.

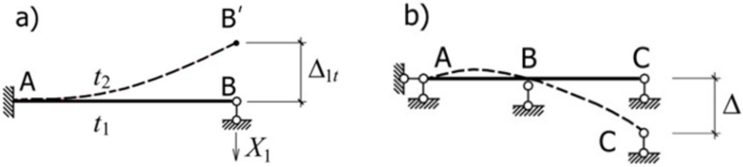


Figure 8.3

If we assume that in the truss (Figure 8.2) the length of the rod 1–4 turned out to be less than the size required by the project, then in order to attach its ends to the nodes, the rod would have to be pulled. This means that the entire group of rods of this panel of the truss will undergo additional forces even before the given load is applied, in particular, rods 1–4 and 2–3 will be stretched, and four other rods will be compressed (the initial stress state arises).

2. The forces in statically indeterminate systems arising from an external load depend on the ratios of the rigidity of the system elements.

Compare, for example, the distribution of bending moments in the frame (Figures 8.4, a, b) with different ratios of bending rigidity of the members.

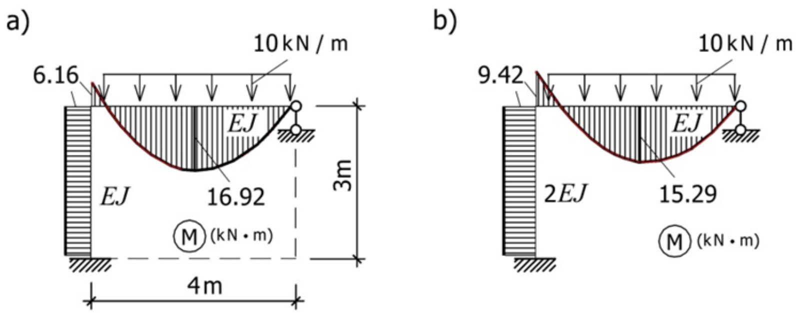


Figure 8.4

The forces in these systems, arising from thermal effects and settlements of supports, depend on the rigidity values of the members.

3. A system with *n* redundant constraints retains geometrical immutability even after the loss of these constraints, while a statically determinate system, after the removal of at least one constraint, turns into a changeable one.

4. The displacements of statically indeterminate systems are, as a rule, less than the corresponding displacements of those statically determinate systems from which they are formed. For example, as follows from the analysis of the work under load of the beams (Figure 8.5), $\Delta_2 > \Delta_1$.

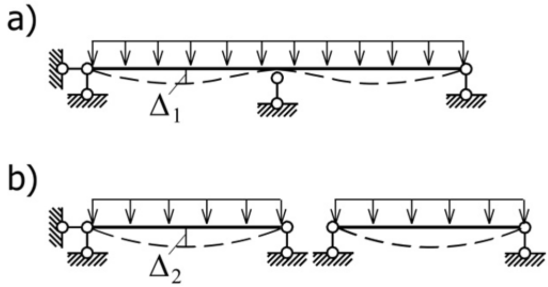


Figure 8.5

Other features of the distribution of forces and displacements in statically indeterminate systems will be explained in the subsequent parts of the chapter.

8.2. Determining the Degree of Static Indeterminacy

By the definition, the degree of static indeterminacy is equal to the number of redundant constraints. From the formula (1.1), which establishes quantitative relations between the number of disks degree of freedom and the number of constraints superimposed on them, it follows that the number of redundant constraints (Λ) will be equal to $\Lambda = -W$, that is, calculated by the formula:

$$\Lambda = L_0 + 2H + 3R - 3D, \qquad (8.1)$$

and if the disks are connected only by constraints of the first (single link) and second (hinge) types, then by the formula:

$$\Lambda = L_0 + 2H - 3D. \tag{8.2}$$

As in the definition of W, both formulas can be used when none of the disks of the system is represented as a closed contour.

If the outline of the frame is closed, it must be divided into several open ones and only then the formula (8.1) should be used.

Hingeless closed contour is three times statically indeterminate. Indeed, in order to turn the frame with the form of a closed contour (Figure 8.6, a) into a statically determinate frame (Figure 8.6, b) it is possible to remove three constraints in the cross-section k. These three links transfer internal forces from one end of the member to the other.

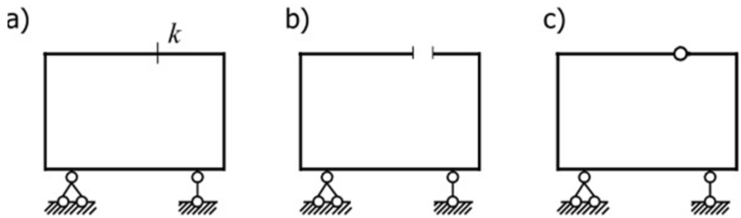


Figure 8.6

If in the cross-section k the constraint will be removed, through which the bending moment is transferred from one part of the member to another, i.e. set the hinge, we get twice statically indeterminate frame (Figure 8.6, c).

Thus, the degree of static indeterminacy of the frame can be determined by the formula:

$$\Lambda = 3K - H, \tag{8.3}$$

where K is the number of closed contours in the frame;

H is the number of simple hinges.

Note that the frame shown in Figure 8.7 also represents a hingeless closed contour. The base, to which the frame is attached at points A and B, in this case, is considered as a disk connecting these points.

Here are some examples. Let us determine the degree of static indeterminacy for the frame shown in Figure 8.8.

By the formula (8.2) we get:

$$\Lambda = L_0 + 2H - 3D = 9 + 2\cdot 2 - 3\cdot 3 = 4\,.$$

By the formula (8.3):

$$\Lambda = 3K - H = 3\cdot 2 - 2 = 4\,.$$

Closed contours are shown in Figure 8.8 by wavy line.

Figure 8.7

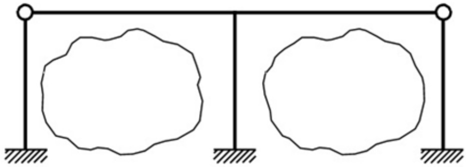


Figure 8.8

When using formula (8.1) for the frame shown in Figure 8.9, we take into account that disks 1 and 2, as well as 2 and 3 are rigidly connected to each other.

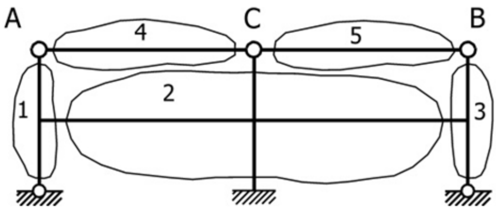


Figure 8.9

The outlines of the discs are highlighted by wavy lines. The hinge at the point C is double one.

$$\Lambda = L_0 + 2H + 3R - 3D = 7 + 2\cdot 4 + 3\cdot 2 - 3\cdot 5 = 6\,.$$

By the formula (8.3) we get:

$$\Lambda = 3K - H = 3\cdot 4 - 6 = 6\,.$$

The partition of the frame (Figure 8.10) into individual disks will be accepted as shown in the figure.

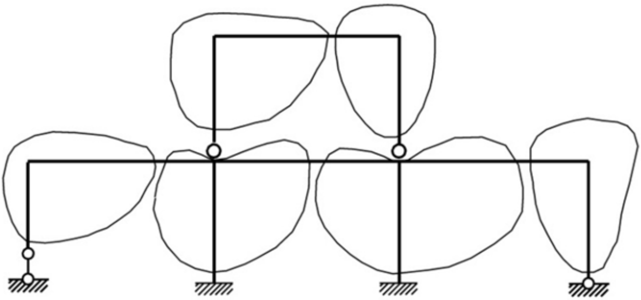


Figure 8.10

Then we get: $D = 6,\ \ H = 2,$ the number of rigid (fixed) connections (nodes) $R = 4.$

By the formula (8.1):

$$\Lambda = L_0 + 2H + 3R - 3D = 9 + 2\cdot 2 + 3\cdot 4 - 3\cdot 6 = 7\ .$$

By the formula (8.3):

$$\Lambda = 3K - H = 3\cdot 4 - 5 = 7\ .$$

In the previous expression it is accepted that $H = 2,$ since there are two simple hinges on the scheme (each of them connects only two disks).

In the last expression $H = 5,$ since in addition to two hinges in the upper contour, two hinges in the lower left contour and one hinge in the lower right contour are taken into account.

The degree of static indeterminacy defines the number of additional equations that need to be written to determine unknown forces. These unknowns are efforts in redundant constraints.

8.3. Primary System and Primary Unknowns

The sequence of actions for disclosing the static indeterminacy of a given system is as follows.

In a given statically indeterminate system, redundant constraints are removed, and unknown forces are applied instead. The obtained system is called the primary system of the force method, and unknown forces are called the primary unknowns of this method. They are designated with symbols X_i, where $i = 1, 2, \cdots, n$ $(n \le \Lambda)$.

In order to reduce the number of unknowns, experienced specialists use sometimes statically indeterminate primary systems. The number of unknowns (n) in this case will be less than the number of redundant constraints (Λ). This calculation method requires additional calculations for statically indeterminable fragments included in the main primary system.

Subsequently, by comparing the displacements of the given and the primary systems, equations are obtained for determining the primary unknowns.

Let us explain some features of the choice of the primary system. First of all, we note that the primary system should be geometrically unchangeable and immovable. For any statically indeterminate frame, several primary systems can be selected. Consider the following example. The degree of static indeterminacy of the frame shown in Figure 8.11, a, is three. Possible variants of the primary systems are shown in Figures 8.11, b–c. In Figure 8.11, b it is shown that as the primary unknowns of the force method, the forces in the support connections of a given frame are taken. According to Figure 8.11, c the primary unknowns are X_1, X_3 (reactions in support connections) and X_2 (interaction forces (moments) between the members adjacent to the hinge). The systems shown in Figures 8.11, d, e, cannot be selected as the primary ones, since they are instantly changeable.

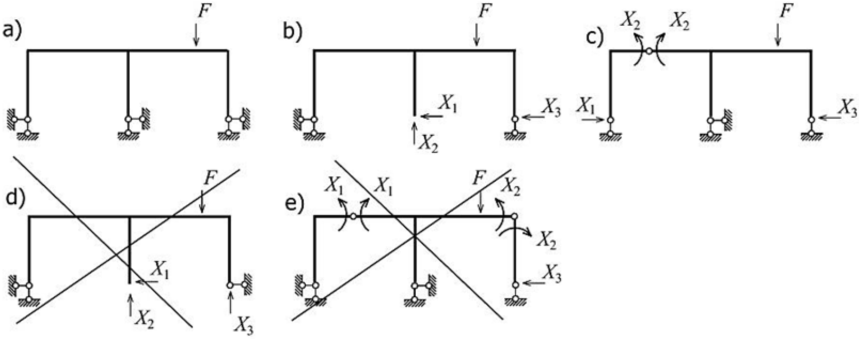


Figure 8.11

All subsequent calculations in the force method are associated with the primary system. Therefore, the complexity of the calculation will substantially depend on the successful choice of a variant of the primary system. Methods for selecting rational primary systems are outlined in Section 8.8.

8.4. Canonical Equations

Deformations of the given and the primary systems will be the same only if the displacements of the application points of the primary unknowns in their directions in the primary system are the same as in the given system, i.e., equal to zero.

Indeed, for example (Figure 8.11, a–c), in the given system the displacement in the direction of force X_1 or X_3 is equal to zero. The angle of mutual rotation of the cross-sections in the direction of unknown X_2 (Figure 8.11, c) is equal to zero also.

The displacements in the primary system in the directions of the primary unknowns depend on the external load acting on the system and the primary unknowns, so we can write that:

$$\left.\begin{array}{l} \Delta_1\left(X_1, X_2, \cdots, X_n, F\right)=0; \\ \Delta_2\left(X_1, X_2, \cdots, X_n, F\right)=0; \\ \cdot \qquad \cdot \qquad \cdot \\ \Delta_n\left(X_1, X_2, \cdots, X_n, F\right)=0. \end{array}\right\} \tag{8.4}$$

where $\Delta_i \left(i=1,\cdots,n\right)$ – is full displacement in the direction of the unknown X_i, that is, displacement caused by the unknowns $X_1, X_2,..., X_n$ and external load F.

The number *n* of such equations certainly corresponds to the number of primary unknowns. If we use the principle of independence of the action of forces, then the *i*-th equation from system (8.4) can be written in the form that allows us to see the contribution of each force factor to the final result:

$$\Delta_i = \Delta_{i1} + \Delta_{i2} + \cdots + \Delta_{in} + \Delta_{iF}, \tag{8.5}$$

where $\Delta_{i1}, \Delta_{i2}, \cdots, \Delta_{in}$ are the displacements of the application point of the i-th primary unknown in its direction, caused by forces $X_1, X_2, \cdots, X_n$;

Δ_{iF} are the displacement of the same point in the same direction, caused by an external load.

The displacement in the direction of the i-th unknown, caused by force X_k, can be represented as:

$$\Delta_{ik} = \delta_{ik} X_k, \tag{8.6}$$

where δ_{ik} – is the displacement in the same direction caused by force $X_k = 1$.

Taking into account expressions (8.5) and (8.6), we write the system of equations (8.4) as follows:

$$\left.\begin{array}{l} \delta_{11}X_1 + \delta_{12}X_2 + \delta_{13}X_3 + \cdots + \delta_{1n}X_n + \Delta_{1F} = 0; \\ \delta_{21}X_1 + \delta_{22}X_2 + \delta_{23}X_3 + \cdots + \delta_{2n}X_n + \Delta_{2F} = 0; \\ . \quad . \quad . \quad . \quad . \quad . \quad . \\ \delta_{n1}X_1 + \delta_{n2}X_2 + \delta_{n3}X_3 + \cdots + \delta_{nn}X_n + \Delta_{nF} = 0. \end{array}\right\} \tag{8.7}$$

These equations are called the canonical equations of the force method for calculating the system on the action of an external load. The essence of the i-th equation is that the displacement of the application point of the unknown X_i in its direction, caused by all unknowns and the external load, is zero.

In the matrix-vector form, system (8.7) can be written more compactly:

$$A\,\vec{X} + \vec{B} = 0, \tag{8.8}$$

where A is matrix of coefficients at unknowns in the canonical equations (pliability matrix of the primary system):

$$A = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} & \cdots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \delta_{23} & \cdots & \delta_{2n} \\ . & . & . & . & . \\ \delta_{n1} & \delta_{n2} & \delta_{n3} & \cdots & \delta_{nn} \end{bmatrix}; \tag{8.9}$$

$\vec{X}$ is vector of unknowns:

$$\vec{X}^T = [X_1 \; X_2 \; X_3 \cdots X_n]; \tag{8.10}$$

$\vec{B}$ is vector of free terms of canonical equations (vector of the load displacements):

$$\vec{B}^T = [\Delta_{1F} \; \Delta_{2F} \cdots \Delta_{nF}]. \tag{8.11}$$

The coefficients of the type δ_{ii}, i.e., located on the main diagonal, are called the main ones (main displacements), and the coefficients δ_{ik}, if $i \neq k$ – are called the secondary ones (secondary displacements). According to the reciprocity theorem $\delta_{ik} = \delta_{ki}$, i.e., the matrix A is symmetric.

When calculating the statically indeterminate system on the thermal effect, the vector $\vec{B}$ in equation (8.8) has the form:

$$\vec{B}^T = [\Delta_{1t} \; \Delta_{2t} \cdots \Delta_{nt}], \tag{8.12}$$

where Δ_{it} is the displacement of the application point of the *i*-th unknown in its direction, caused by a change in the temperature of the members.

When calculating the system by the settlements of supports:

$$\vec{B}^T = [\Delta_{1c} \; \Delta_{2c} \cdots \Delta_{nc}], \tag{8.13}$$

where Δ_{ic} is the displacement in the direction of the *i*-th unknown caused by the settlements of supports.

8.5. Determining Coefficients and Free Terms of Canonical Equations

The coefficients and free terms of the canonical equations are calculated according to the rules of determining displacements described in Chapter 7. For the frame systems that experience predominantly bending deformations in non-automated computing (“manual” calculation), we can take into account the influence on displacements of only bending moments. Therefore, displacements δ_{ik} and Δ_{iF} are calculated by the formulas:

$$\delta_{ik} = \sum \int \frac{\bar{M}_i \bar{M}_k \, dx}{EJ},$$

$$\Delta_{iF} = \sum \int \frac{\bar{M}_i M_F \, dx}{EJ},$$

where $\bar{M}_i$, $\bar{M}_k$ are bending moments diagrams caused by dimensionless forces, respectively $X_i = 1$ and $X_k = 1$;

M_F is bending moments diagram caused by external load.

So, for example, if for the frame (Figure 8.12, a) we accept the primary system according to the variant of Figure 8.12, b, when determining the displacement δ_{21}, it is necessary to consider the state of the frame under the action $X_1 = 1$ (Figure 8.12, c) as load state, and the second state, corresponding to the action $X_2 = 1$ (Figure 8.12, d), as an auxiliary one. Then, after the construction of the bending moments diagram (Figures 8.12, f, g), you can use the well-known methods of calculating the Mohr integral of the form:

$$\delta_{21} = \sum \int \frac{\bar{M}_2 \bar{M}_1 \, dx}{EJ}.$$

Displacement Δ_{1F} (Figure 8.12, e) is calculated using diagrams $\bar{M}_1$ (Figure 8.12, e) and M_F (Figure 8.12, h):

$$\Delta_{1F} = \sum \int \frac{\bar{M}_1 M_F \, dx}{EJ}.$$

The matrix form for determining displacements is described in Section 7.10.

Obviously, the values of the coefficients and free terms of the canonical equations are more accurate if in addition to bending moments we take into account the longitudinal and shear forces in the frame elements.

After determining the coefficients and free terms, the system of canonical equations can be solved in numerical form.

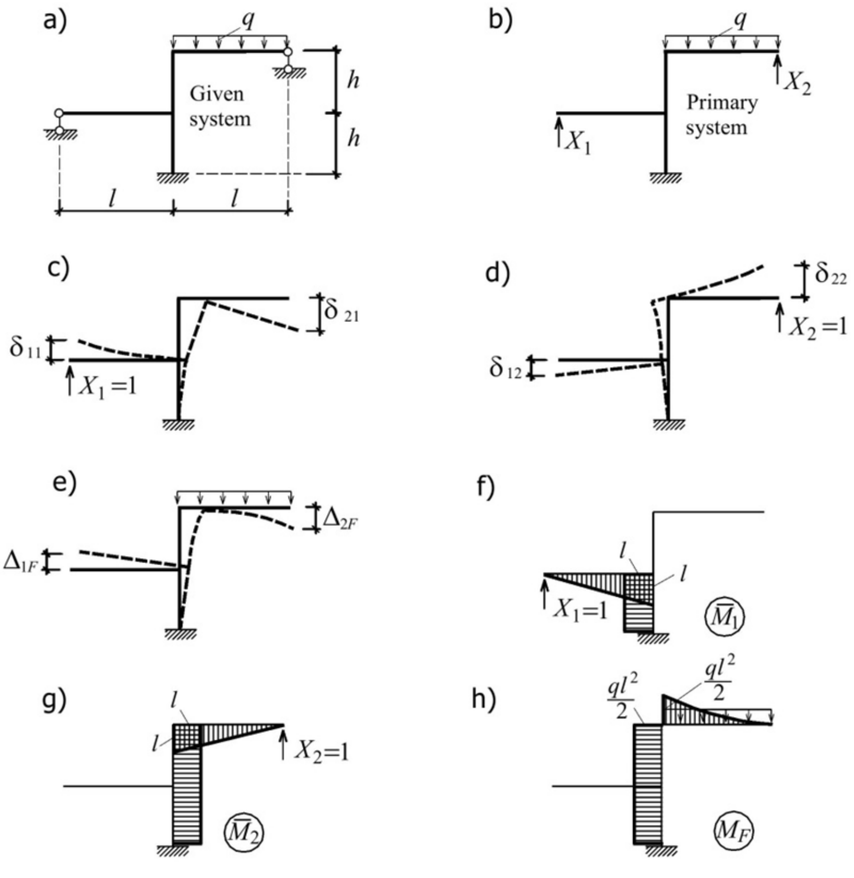


Figure 8.12

8.6. Constructing the Final Diagrams of the Internal Forces

The solution of the system of canonical equations allows us to find the values of the primary unknowns. The final efforts $(S \in \{M, Q, N\})$ in the k-th cross-section of a given system are calculated by the expression, based on the principle of independence of the forces action:

$$S_k = S_{kF} + \bar{S}_{k1} X_1 + \bar{S}_{k2} X_2 + \cdots + \bar{S}_{kn} X_n , \tag{8.14}$$

where S_{kF} is the force in the k-th section from the action of external load;

$\bar{S}_{ki}$ is the force in k-th section from $X_i = 1,\ i = 1,\ 2, \cdots, n$.

In accordance with expression (8.14), the final diagrams of bending moments, shear and longitudinal forces are constructed:

$$M = M_F + \bar{M}_1 X_1 + \bar{M}_2 X_2 + \cdots + \bar{M}_n X_n , \tag{8.15}$$

$$Q = Q_F + \bar{Q}_1 X_1 + \bar{Q}_2 X_2 + \cdots + \bar{Q}_n X_n ,$$

$$N = N_F + \bar{N}_1 X_1 + \bar{N}_2 X_2 + \cdots + \bar{N}_n X_n .$$

Constructing diagrams Q and N using the above formulas is not always convenient. A simpler way of constructing the diagram Q is based on the use of differential dependence $Q = \dfrac{dM}{dx}$.

To use this dependence we obtain an analytical expression for determining the bending moment in the cross-section of a frame member. Consider such a member as a beam on two supports. Suppose that the beam at its span is loaded as shown in Figure 8.13, a. Both external moments at the supports (left (l) and right (r)) cause in the cross-sections of the beam over the supports the positive bending moments equel to M^l and M^r.

Having constructed for this beam the moment diagrams caused by span load (Figure 8.13, b) and supporting moments (Figure 8.13, c, d), we will determine, based on the principle of independence of the action of forces, the final ordinate in the cross-section k on the diagram M as the sum of its components :

$$M = M_F + \frac{l - x}{l} M_l + \frac{x}{l} M_r . \tag{8.16}$$

Taking the first derivative of the expression (8.16), we obtain the formula for determining the shear force in the same cross-section:

$$Q = Q_F + \frac{M_r - M_l}{l}. \tag{8.17}$$

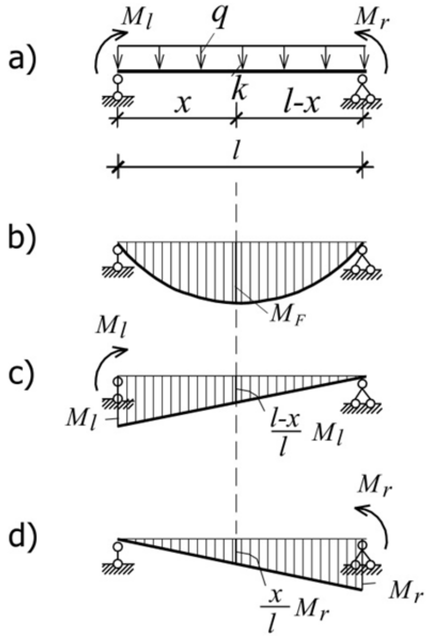


Figure 8.13

8.7. Calculation Algorithm. Calculation Check

The process of calculating statically indeterminate frames by the force method includes the following steps.

1. Determination of the degree of static indeterminacy of the system.
2. Selection of the primary system.
3. The recording of the system of canonical equations in the general form.
4. Construction of the diagrams of the internal forces in the primary system due to the external load and the unit values of the primary unknowns.

5. Calculation of the coefficients at the unknown and free terms of the canonical equations.

6. Recording the system of canonical equations in numerical form and solving it.

7. Construction of the final diagram of bending moments.

8. Construction of the final diagrams of Q and N.

In order not to be mistaken during the calculation, the calculations at each step of the algorithm should be checked. For this, of course, it is necessary to understand thoroughly the essence of the operations performed and correctly use the knowledge accumulated during the studying the course of structural mechanics.

Let us explain the features of checking the accuracy of the calculation at individual steps of the algorithm.

First of all, we make a remark on the question of choosing the primary system. For all possible variants of the primary system, a kinematic analysis of them should be performed in the sequence recommended in Chapter 1. Particular attention should be paid to the analysis of the structure of the system and its verification for instantaneous changeability.

At the step of constructing the efforts diagrams in the primary system, as a rule, the static method is used. To check the diagrams, the conditions of equilibrium of fragments of the design scheme, in particular, frame nodes, are used the most.

Verification of the calculation of the coefficients at the unknown and free terms of the canonical equations is carried out using the total diagram of the unit moments M_s, construct according to the rule:

$$\bar{M}_s = \bar{M}_1 + \bar{M}_2 + \cdots + \bar{M}_n . \tag{8.18}$$

If we "multiply" diagram M_i and diagram M_s, we get:

$$\begin{aligned} \delta_{is} &= \sum\int \frac{\bar{M}_i \bar{M}_s \, dx}{EJ} = \sum\int \frac{\bar{M}_i \left(\bar{M}_1 + \bar{M}_2 + \cdots + \bar{M}_n\right) dx}{EJ} = \\ &= \sum\int \frac{\bar{M}_i \bar{M}_1 \, dx}{EJ} + \sum\int \frac{\bar{M}_i \bar{M}_2 \, dx}{EJ} + \cdots + \sum\int \frac{\bar{M}_i \bar{M}_n \, dx}{EJ} = \\ &= \delta_{i1} + \delta_{i2} + \cdots + \delta_{in} = \sum \delta_{ik}, \; k = 1, 2, \cdots, n, \end{aligned} \tag{8.19}$$

i.e., the sum of the coefficients for unknowns in the i-th $(i = 1, 2, \cdots, n)$ equation should be equal δ_{is}. Such a check is called line by line.

Instead of "multiplying" each unit moment diagram by total diagram M_s, in practice, we can "multiply" $\bar{M}_s$ by $\bar{M}_s$. Using (8.19), it is easy to show that:

$$\delta_{ss} = \sum \int \frac{\bar{M}_s \bar{M}_s \, dx}{EJ} = \sum_{i=1}^{n} \sum_{k=1}^{n} \delta_{ik} , \tag{8.20}$$

i.e. δ_{ss} equal to the sum of all the coefficients of the canonical equations.

This check is called universal.

Similarly, verification of the calculation of free terms is performed:

$$\Delta_{sF} = \sum \int \frac{\bar{M}_s M_F \, dx}{EJ} = \sum_{i=1}^{n} \Delta_{iF} . \tag{8.21}$$

The sum of all free terms of the canonical equations is Δ_{sF}.

It should be noted that performing the checks of coefficients and free terms mentioned here is not always a guarantee of correct calculations. In the course of determining δ_{ik}, Δ_{iF} and δ_{ss}, Δ_{sF} in some step, the same mistake can be made and, as a result, it will be unnoticed. Therefore, we recall once again that the basis of correct calculations at this step is knowledge and the ability to apply methods for calculating the Mohr integrals correctly.

To verify the final diagrams of bending moments static and kinematic checks are used. The static check of the diagram "*M*" carries out by checking the equilibrium of the frame nodes. With its help, only errors that can be made during the step of constructing the bending moment diagram using the formula (8.15) are detected.

The main verification is kinematic one (its other names: deformation check, check of displacements). The displacement of the application point of the *i*-th primary unknown in its direction in the given system should be equal to zero. Therefore, using the general rule for determining displacements, we obtain:

$$\sum \int \frac{\bar{M}_i M \, dx}{EJ} = 0 . \tag{8.22}$$

In this case, it is clear that the sum of the displacements along the directions of all the primary unknowns is also equal to zero. Consequently,

$$\sum\int\frac{\bar{M}_s\, M\, dx}{EJ}=0, \tag{8.23}$$

i.e., the result of "multiplying" the total unit diagram $\bar{M}_s$ by the final diagram of the moments must be equal to zero.

The static check of the diagrams Q and N consists in checking the equilibrium of the part of the frame cut off from the support connections.

E x a m p l e. Construct the diagrams M, Q and N for the frame shown in Figure 8.14, a.

The given frame is twice statically indeterminate. The primary system and the primary unknowns are shown in Figure 8.14, b. The system of canonical equations has the form:

$$\left.\begin{aligned}\delta_{11}X_1+\delta_{12}X_2+\Delta_{1F}&=0;\\ \delta_{21}X_1+\delta_{22}X_2+\Delta_{2F}&=0.\end{aligned}\right\}$$

Diagrams of bending moments in the primary system caused by the action of $X_1=1$, $X_2=1$ and external load are shown in Figures 8.14, c, d, e.

We determine the coefficients at unknowns and the free terms in the canonical equations:

$$\delta_{11}=\frac{1}{2EJ}\left(\frac{1}{2}1\cdot1\frac{2}{3}1+\frac{1}{2}3\cdot3\frac{2}{3}3\right)+\frac{1}{EJ}3\cdot6\cdot3+\frac{1}{EJ}\frac{1}{2}3\cdot3\frac{2}{3}3=\frac{203}{3EJ};$$

$$\delta_{22}=\frac{1}{2EJ}6\cdot4\cdot6+\frac{1}{EJ}\frac{1}{2}6\cdot6\frac{2}{3}6=\frac{144}{EJ};$$

$$\delta_{12}=\delta_{21}=-\frac{1}{2EJ}6\cdot4\cdot1-\frac{1}{EJ}3\cdot6\cdot3=-\frac{66}{EJ};$$

$$\Delta_{1F}=\frac{1}{2EJ}320\cdot4\cdot1+\frac{6}{6EJ}\left(320\cdot3+4\cdot125\cdot3+20\cdot3\right)=\frac{3160}{EJ};$$

$$\Delta_{2F}=-\frac{1}{2EJ}320\cdot4\cdot6+\frac{6}{6EJ}\left(-320\cdot6-4\cdot125\cdot3\right)=-\frac{7260}{EJ}.$$

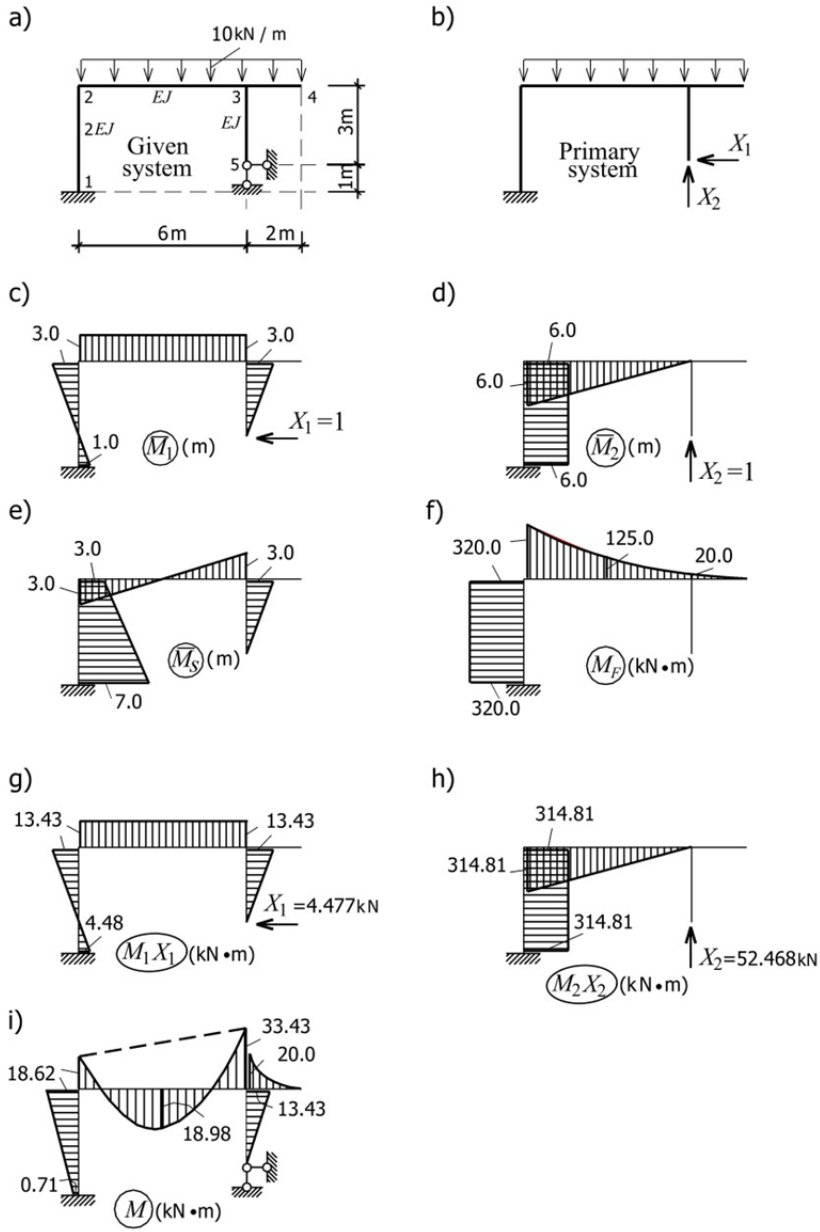


Figure 8.14 (begining)

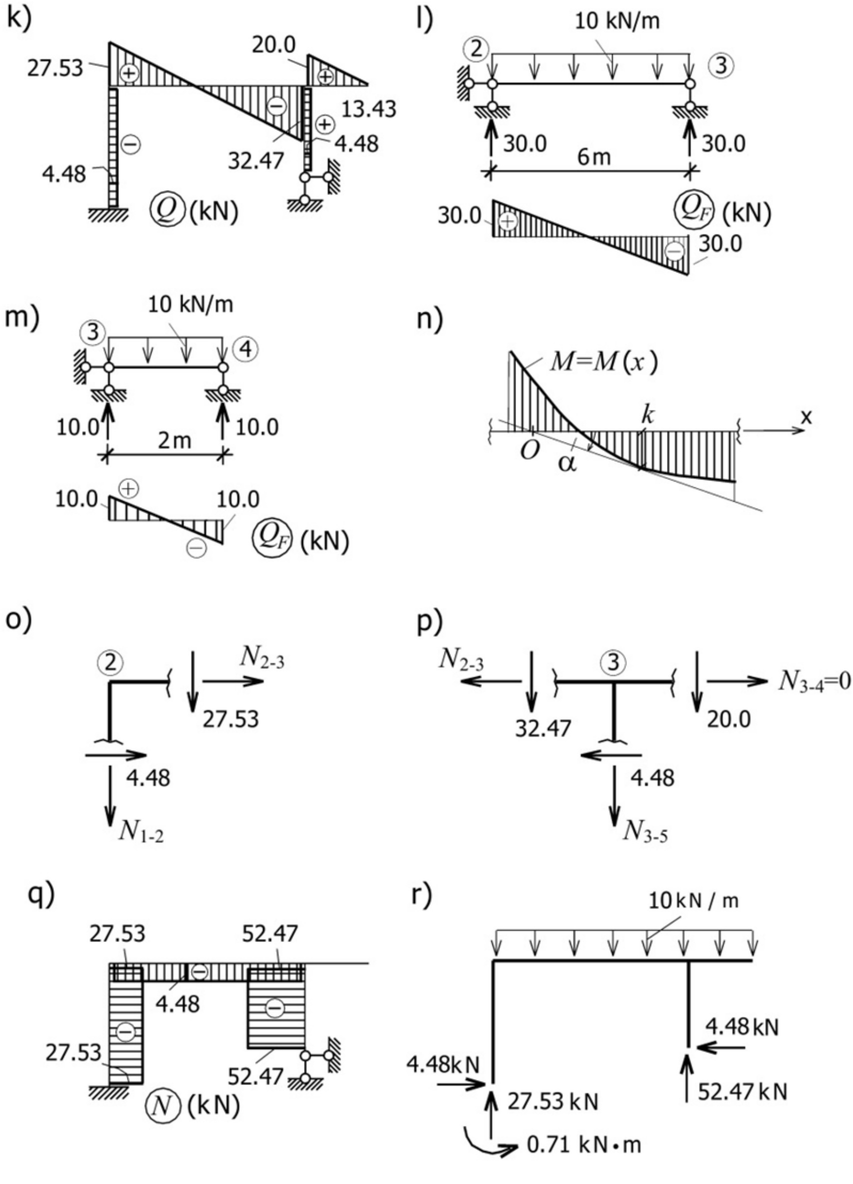


Figure 8.14 (ending)

To check the coefficients and free terms, a total diagram of the unit moments is constructed (Figure 8.14, e). Using the formula (8.20), we obtain:

$$\delta_{ss}=\frac{4}{6\cdot 2EJ}(2\cdot 3\cdot 3+2\cdot 7\cdot 7+3\cdot 7\cdot 2)+$$
$$+\frac{6}{6EJ}(3\cdot 3+3\cdot 3)+\frac{1}{EJ}\frac{1}{2}3\cdot 3\frac{2}{3}3=\frac{239}{3EJ}.$$

Indeed:

$$\delta_{11}+\delta_{12}+\delta_{21}+\delta_{22}=\frac{203}{3EJ}-\frac{66}{EJ}-\frac{66}{EJ}+\frac{144}{EJ}=\frac{239}{3EJ}.$$

By the formula (8.21) we have:

$$\Delta_{sF}=-\frac{1}{2EJ}320\cdot 4\cdot 5+\frac{6}{6EJ}\left(-320\cdot 3+20\cdot 3\right)=-\frac{4100}{EJ},$$

that is equal to $\Delta_{1F}+\Delta_{2F}=\frac{3160}{EJ}-\frac{7260}{EJ}=-\frac{4100}{EJ}$.

We record the system of equations in numerical form:

$$\left.\begin{aligned}\frac{203}{3EJ}X_1-\frac{66}{EJ}X_2+\frac{3160}{EJ}=0;\\ -\frac{66}{EJ}X_1+\frac{144}{EJ}X_2-\frac{7260}{EJ}=0.\end{aligned}\right\}$$

Having solved this system of equations, we find:

$$X_1=4.477\,\text{kN};\qquad X_2=52.468\,\text{kN}.$$

To construct the final moment diagrams, we use the formula (8.15). The diagrams M_1X_1 and M_2X_2 are shown in Figures 8.14, g, h, and the final diagram M is shown in Figure 8.14, i. Its static verification is performed (The reader is advised to conduct its own verification). We perform a kinematic check:

$$\sum\int\frac{\bar{M}_s M\,dx}{EJ}=\frac{4}{6\cdot 2EJ}\left(-2\cdot 3\cdot 18.62-2\cdot 7\cdot 0.71-3\cdot 0.71-7\cdot 18.62\right)+$$

$$+\frac{6}{6EJ}\left(-3\cdot 18.62+3\cdot 33.43\right)+\frac{1}{EJ}\frac{1}{2}3\cdot 3\frac{2}{3}13.43=$$

$$=-\frac{140.57}{EJ}+\frac{140.55}{EJ}=-\frac{0.02}{EJ}.$$

The relative error of the calculations is:

$$\varepsilon=\left|\frac{-0.02}{140.55}\right|\cdot 100\approx 0.01\,\%$$

which is less than the acceptable value.

The diagram Q (Figure 8.14, k) is constructed in accordance with the diagram M. Once again, we note that a simpler way of constructing is based on dependency

$$Q=\frac{dM}{dx}.$$

We use the formula (8.17).

Considering the element 2–3 as a simple beam loaded with a uniformly distributed load, we construct a diagram of the shear forces (diagram of the shear forces for the beam). It is shown in Figure 8.14, l.

Given the distribution of moments on this element (Figure 8.14, i) using the formula (8.17), we find that in the cross-section adjacent to the node 2:

$$Q_2=30+\frac{-33.43-\left(-18.62\right)}{6}=27.53\ \text{kN},$$

and in the cross-section adjacent to the node 3:

$$Q_3=-30+\frac{-33.43-\left(-18.62\right)}{6}=-32.47\ \text{kN}.$$

The Q diagram for the cantilever 3–4 is constructed as for a statically determinable fragment of a frame. However, in this case we can also use the formula (8.17), if you consider section 3–4 as a beam with two supports (Figure 8.14, m).

Then in the cross-section adjacent to the node 3:

$$Q_3 = 10 + \frac{0-(-20)}{2} = 20 \text{ kN},$$

and in the cross-section adjacent to the node 4:

$$Q_4 = -10 + \frac{0-(-20)}{2} = 0.$$

For element 1–2 we get:

$$Q_1 = 0 + \frac{-18.62-(-0.71)}{4} = -4.48 \text{ kN},$$

$$Q_2 = 0 + \frac{-18.62-(-0.71)}{4} = -4.48 \text{ kN}.$$

Mind that $\frac{dM}{dx} = \operatorname{tg}\alpha$. The diagram of bending moments is usially constructed on the stretched fibers of the element. For horizontal elements, the positive ordinates of the bending moments must be located below the axis of the element. Therefore, the sign of the transverse force in a given cross-section "*k*" of the horizontal bar can be determined as follows. Drawing a tangent to the line bounding the diagram M, at a point, corresponding to the position of the cross-section k (Figure 8.14, n), it is necessary to find the intersection point of this tangent and the axis of the element (point O).

If the axis of the element must be rotated around the point O until it coincides with the tangent in the shortest way clockwise, then the shear (transverse) force in the cross-section k will be positive $(Q > 0)$. When the axis of the element moves anticlockwise, then the shear force in the cross-section will be negative $(Q<0)$.

On the linear zones of the diagram of bending moments, the position of the tangent coincides with the line bounding the diagram. The shear force along the entire length of this section will be constant. For the element 3–5

$$Q=\frac{13.43}{3}=-4.48 \text{ kN},$$

and for the element 1–2

$$Q=-\frac{18.62-0.71}{4}=-4.48 \text{ kN}.$$

After determining the shear forces in the frame elements the longitudinal forces N are determined from the equilibrium equations of the nodes. The calculations begin with a node in which the elements with no more than two unknown forces are joined, and then, sequentially cutting out the nodes, determine the efforts in all other bars. The equilibrium equations are written as the sum of the projections of all the forces (both internal and external forces applied to the nodes, if any) on the vertical and horizontal axes. In the presence of inclined bars, if the calculations may be simplified, the forces projections can be performed to the axes perpendicular to the bars directions.

Composing equations for node 2 (Figure 8.14, o) $\sum X=0$, $\sum Y=0$, we find $N_{2-3}=-4.48$ kN, $N_{1-2}=-27.53$ kN.

From the equation $\sum Y=0$ for node 3 (Figure 8.14, p) we get $N_{3-5}=-52.47$ kN.

The equation $\sum X=0$ for node 3 is a test one. The diagram N is shown in Figure 8.14, p.

For carrying out a static check of the diagrams Q and N we cut off the frame from the support connections, load it by a given load and shear and longitudinal forces in the cross-sections separating the rods from the support connections (Figure 8.14, c). Composing the equations $\sum X=0$, $\sum Y=0$ and $\sum M=0$, we make sure that the frame is in equilibrium.

8.8. The Concept of Rational Primary System and Methods of Its Choice

A rational primary system is such a system for which in the canonical equations greatest possible number of secondary coefficients is zero. At the same time, it is very important to set zero coefficients only on the basis of a visual analysis of the outline of the force diagrams, without spending time on their numerical determination. Zeroing secondary coefficients leads to significant simplifications in the calculation.

If some coefficient δ_{ik} is equal to zero, the corresponding diagrams $\bar{M}_i$ and $\bar{M}_k$ are usually called mutually orthogonal. An analogy with the scalar product of mutually orthogonal vectors is used.

The most commonly used methods for obtaining rational primary systems include: using the symmetry of the system, grouping unknowns, transforming of the load, breaking up multi-span frames.

1. Using the symmetry of the system. The primary system for a frame which has a symmetric geometric dimensions and symmetric rigidity of the elements should be taken symmetrical. If the primary unknowns can be positioned on the axis of symmetry, then some of them will be symmetric, and the other – inversely symmetric (or skew-symmetric). Due to the action of a symmetrical load on the symmetrical frame, the distribution of forces in its elements will be symmetric, and vice versa: inverse-symmetrical loading of the symmetrical frame causes inverse-symmetrical forces in its elements. Therefore, the diagrams of bending moments in the primary system will be either symmetrical or inversely-symmetrical. Symmetrical and inverse-symmetrical diagrams are mutually orthogonal.

For example, taking for the frame (Figure 8.15, a) the primary system shown in Figure 8.15, b, we obtain symmetrical diagrams $\bar{M}_1$, $\bar{M}_2$, $\bar{M}_4$ (Figures 8.15, c, d, f) and inverse-symmetrical $\bar{M}_3$ (Figure 8.15, e). Therefore, the coefficients δ_{13}, δ_{31}, δ_{23}, δ_{32}, δ_{34}, δ_{43} are equal to zero.

Crossing out in the system of equations (the reader should write them down) the terms including the listed coefficients, we see that it has decomposed into a subsystem containing only symmetrical unknowns and one equation with inverse-symmetrical unknown.

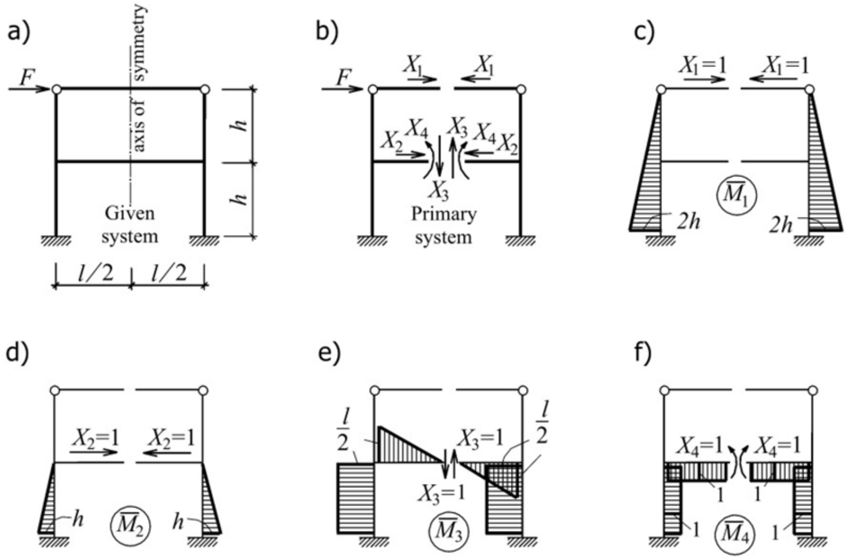


Figure 8.15

It is easy, obviously, to extend the above reasoning to examples of frames with a large number of unknowns.

2. Groupings of the unknown. In many cases, the primary unknowns cannot be positioned on the axis of symmetry. So, for the frame shown in Figure 8.16, a, the number of redundant constraints is six. The symmetric primary system can be adopted according to the variant shown in Figure 8.16, b. However, in this case, when loading it with forces $X_i = 1$ none of the diagrams of bending moments will turn out to be symmetrical or inversely-symmetrical, which means that all secondary coefficients will be nonzero.

In order to obtain symmetrical and invers-symmetrical force plots, it is necessary to introduce new ones (we will denote them Z_i), which are groups of forces, instead of traditional unknowns X_i. The transition from old unknowns to new ones, and vice versa, should be univocal.

In Figure 8.16, c the same primary system with new unknowns is shown. Comparing the location of the unknowns in Figures 8.16, b, c, we find the rules for converting them: each pair of symmetrically located unknowns X_i corresponds to the operation of addition or subtraction of symmetrical and inverse-symmetrical group unknowns Z_i.

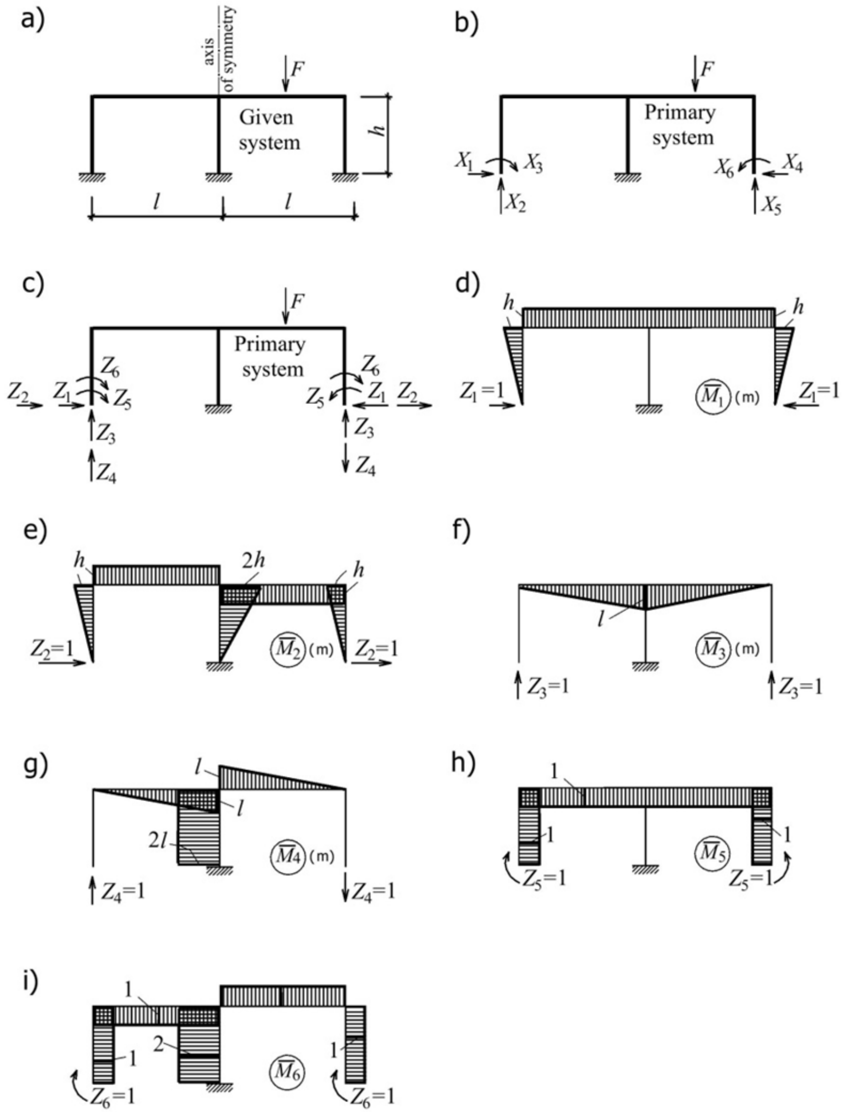


Figure 8.16

In particular, $X_1 = Z_1 + Z_2$, $X_4 = Z_1 - Z_2$, from where the expressions for Z:

$$Z_1 = \frac{X_1 + X_4}{2}, \qquad Z_2 = \frac{X_1 - X_4}{2}.$$

The diagrams of efforts caused by group unknowns are shown in Figure 8.16, d–i. Due to the mutual orthogonality of symmetrical and inverse-symmetrical diagrams, the system of canonical equations decomposes into two independent ones: one of them will include only symmetrical unknowns Z_1, Z_3, Z_5, and the other will include only inverse-symmetrical Z_2, Z_4, Z_6.

3. Transforming of the load. Further simplifications in the calculation of symmetric systems (Figure 8.17, a) are associated with the decomposition of the load into symmetrical and inverse-symmetrical components.

Using the property of mutual orthogonality of the diagrams, it is easy to show that, when a symmetrical load is applied to a symmetrical system, inverse-symmetrical unknowns become zero, and when a inverse-symmetrical load acts, symmetrical unknowns turn out to be zero. In relation to the design scheme of the frame shown in Figure 8.17, b, this means that it should be calculated as systems with three unknowns X_1, X_2, X_4 (the primary system is shown in Figure 8.15, b), and the frame calculation for the action of inverse-symmetrical load (Figure 8.17, c) as systems with one unknown X_3.

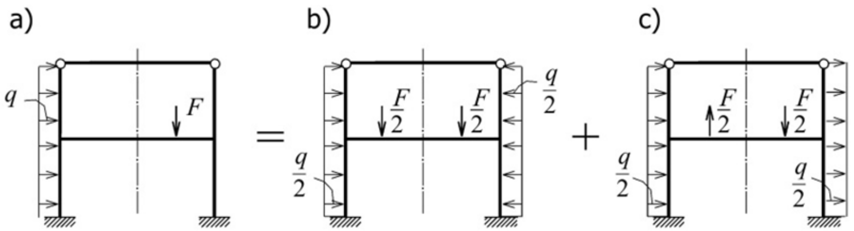


Figure 8.17

4. Breaking up multi-span frames. This method is used for both symmetrical and asymmetrical frames. Less computational work to define $\delta_{i\kappa}$, will be if the diagrams of the internal forces in the primary system extend to small fragments of the frame, i. e., they are "localized" in the vicinity of the load.

For a frame (Figure 8.18, a) with four unknowns in Figures 8.18, b, c, two variants of the primary system are presented. Analyzing the distribution of bending moments due to $X_i = 1$ in the frame shown in Figure 8.18, b, we can verify that none of the coefficients $\delta_{i\kappa}$ is equal to zero.

In the system shown in Figure 8.18, c, bending moment diagrams occur only on columns directly perceiving the action $X_i = 1$. Therefore, $\delta_{13} = \delta_{31} = 0$, $\delta_{14} = \delta_{41} = 0$, $\delta_{24} = \delta_{42} = 0$, and the primary system is rational.

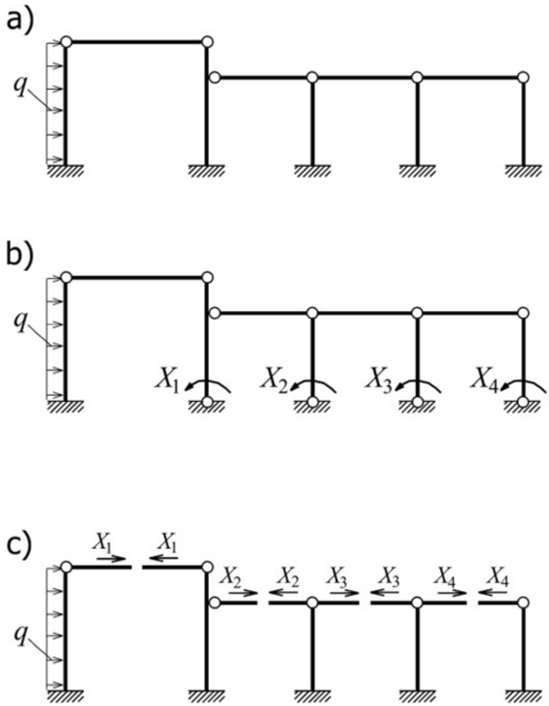


Figure 8.18

8.9. Determining Displacements in Statically Indeterminate Systems

To determine the displacements using the Mohr formula, described in section 7.6, it is necessary to construct in the system the bending moment diagrams caused by the given loading (Figure 8.19, a) and the auxiliary loading (Figure 8.19, b). Then the required displacement will be calculated by the formula (8.24):

$$\Delta_{kF} = \sum\int \frac{\bar{M}_k\, M\, dx}{EJ}. \tag{8.24}$$

However, this method of calculation is not entirely convenient, since it will be necessary to calculate the statically indeterminate system twice.

A simpler calculation method can be obtained from the following reasoning. If you load the primary system with a given load and primary unknowns, which have been determined from the canonical equations, then the diagram of bending moment in this statically determinate system (Figure 8.19, c) will completely coincide with the final moment diagram (Figure 8.19, a). Therefore, if we consider the state of the frame in Figure 8.19, c as the initial one, then to determine the displacement of the point *k* it is possible to take a statically determinate system (Figure 8.19, d) as an auxiliary state. In this case:

$$\Delta_{kF} = \sum\int \frac{\bar{M}_k^0\, M\, dx}{EJ}, \tag{8.25}$$

where $\bar{M}_k^0$ – is the bending moments in a statically determinate system due to $F_k = 1$.

Another method can be used to calculate the same displacement: the diagram of bending moments caused by given load can be constructed in the primary system, and the diagram caused by $F_k = 1$ – in a given statically indeterminate system. We will show this.

Applying reciprocity theorem to the states of the frame shown in Figures 8.19, a, b, we get:

$$F_k\, \Delta_{kF} = F\, \Delta_{Fk}, \tag{8.26}$$

where $F_k = 1$;

F are the forces acting in the state a (this force is a uniformly distributed load q in Figure 8.19, a);

Δ_{Fk} is the displacement caused by $F_k = 1$ in the direction of force F, (in this example, the area of the diagram of vertical displacements of the horizontal element).

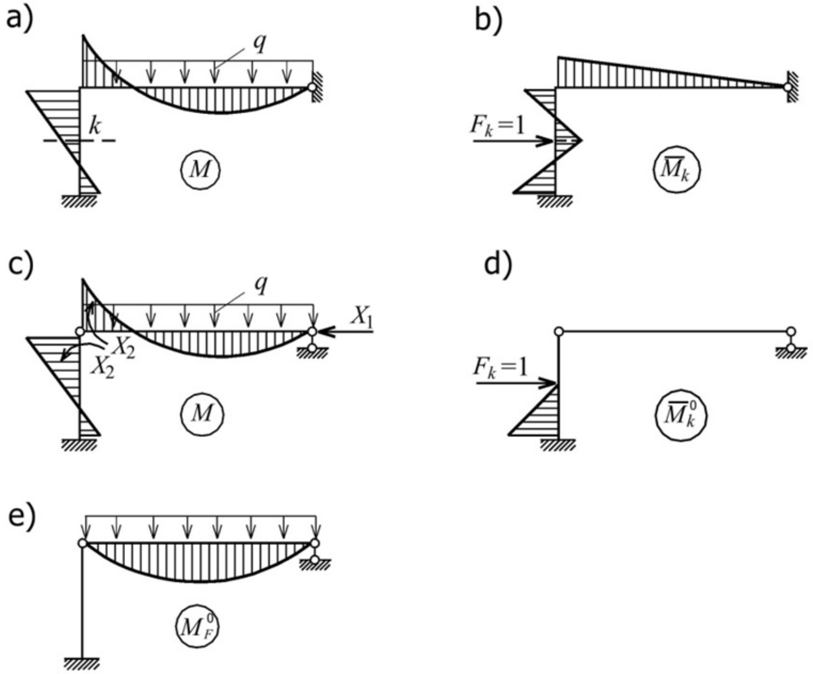


Figure 8.19

Since the diagrams in the states a (Figure 8.19, a) and c (Figure 8.19, c) coincide completely, the expression (8.26) is applicable to the frame states b (Figure 8.19, b) and c. In this case, as F, in Figure 8.19, c, the distributed load and the primary unknowns X_1 and X_2 are accepted. But the work of the primary unknowns on the displacements of the frame in the state b is equal to zero. Therefore:

$$\Delta_{kF} = \sum F \, \Delta_{Fk}, \tag{8.27}$$

i. e., the right side of the expression (8.27) is the work of external forces applied to the primary system. This work is done on the displacements of a statically indeterminate system in state k.

Note that in the above explanations, there were no restrictions on the choice of the primary system.

Writing the expression (8.27) through the work of bending moments, we obtain:

$$\Delta_{kF} = \sum \int \frac{\bar{M}_k \, M_F^0 \, dx}{EJ}, \tag{8.28}$$

where M_F^0 is the bending moments diagram in the primary system (Figure 8.19, e).

Thus, when determining displacements in statically indeterminate systems, one of the "multiplied" diagrams can be built in a given statically indeterminate system, and the second – in any statically determinate one obtained from a given system.

Let us turn to the calculations. In Figure 8.20, a diagram of bending moments in a statically indeterminate frame caused by a given load is shown, and in Figure 8.20, b – diagram of bending moments in the same frame caused by $F_k = 1$. By the formula (8.24) we get:

$$\Delta_{kF} = \sum \int \frac{\bar{M}_k \, M \, dx}{EJ} =$$

$$= \frac{l}{24EJ}\left[-2\frac{15l}{176}\frac{ql^2}{22} - 2\frac{13l}{176}\frac{ql^2}{44} + \frac{15l}{176}\frac{ql^2}{44} + \frac{13l}{176}\frac{ql^2}{22}\right] +$$

$$+ \frac{l}{24EJ}\left[2\frac{3l}{176}\frac{ql^2}{11} - 2\frac{13l}{176}\frac{ql^2}{44} - \frac{13l}{176}\frac{ql^2}{11} + \frac{3l}{176}\frac{ql^2}{44}\right] +$$

$$+ \frac{l}{6EJ}\left[\frac{3l}{176}\frac{ql^2}{11} - 4\frac{3l}{352}\frac{7}{88}ql^2\right] = -\frac{ql^4}{1408}\frac{1}{EJ} \text{ м.}$$

In Figure 8.20, c the diagram of moments in a statically determinate frame (primary system) caused by $F_k = 1$ is shown, and in Figure 8.20, d –

plot of moments in the primary system caused by a given load. By the formula (8.25) we get:

$$\Delta_{kF} = \sum\int \frac{\bar{M}_k^0\, M\, dx}{EJ} = \frac{l}{24EJ}\left[-2\frac{l}{4}\frac{ql^2}{22} + \frac{l}{4}\frac{ql^2}{44}\right] = -\frac{ql^4}{1408}\frac{1}{EJ}\ \text{m}.$$

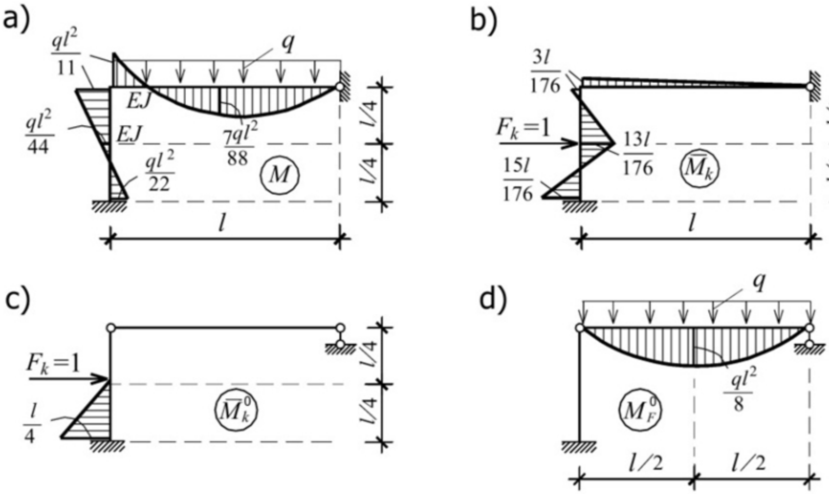


Figure 8.20

According to the formula (8.28):

$$\Delta_{kF} = \sum\int \frac{\bar{M}_k\, M_F^0\, dx}{EJ} = -\frac{1}{EJ}\frac{2}{3}\frac{ql^2}{8}\,l\,\frac{1}{2}\frac{3l}{176} = -\frac{ql^4}{1408}\frac{1}{EJ}\ \text{m}.$$

It is clear that the calculations of displacements using formulas (8.25) or (8.28) are simpler than using the formula (8.24).

8.10. Calculating Frames Subjected to Change of Temperature and to Settlement of Supports

When calculating the frames subjected to the thermal effect, the canonical equations of the force method are recorded in the form:

$$\left.\begin{array}{l}\delta_{11}X_1+\delta_{12}X_2+\delta_{13}X_3+\cdots+\delta_{1n}X_n+\Delta_{1t}=0;\\ \delta_{21}X_1+\delta_{22}X_2+\delta_{23}X_3+\cdots+\delta_{2n}X_n+\Delta_{2t}=0;\\ \cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot\\ \delta_{n1}X_1+\delta_{n2}X_2+\delta_{n3}X_3+\cdots+\delta_{nn}X_n+\Delta_{nt}=0.\end{array}\right\}$$

To calculate the free terms of the equations, formula (7.12) is used.

In statically determinate systems, there are not the efforts caused by the action of the temperature. Therefore, the final diagram of bending moments in a given frame is constructed by summing up unit diagrams of moments multiplied by found from the equations values of corresponding unknowns:

$$M=\bar{M}_1X_1+\bar{M}_2X_2+\cdots+\bar{M}_nX_n\,. \tag{8.29}$$

Kinematic check comes down to the verification of the frame displacements in the direction of redundant constraints, i. e. checking the condition:

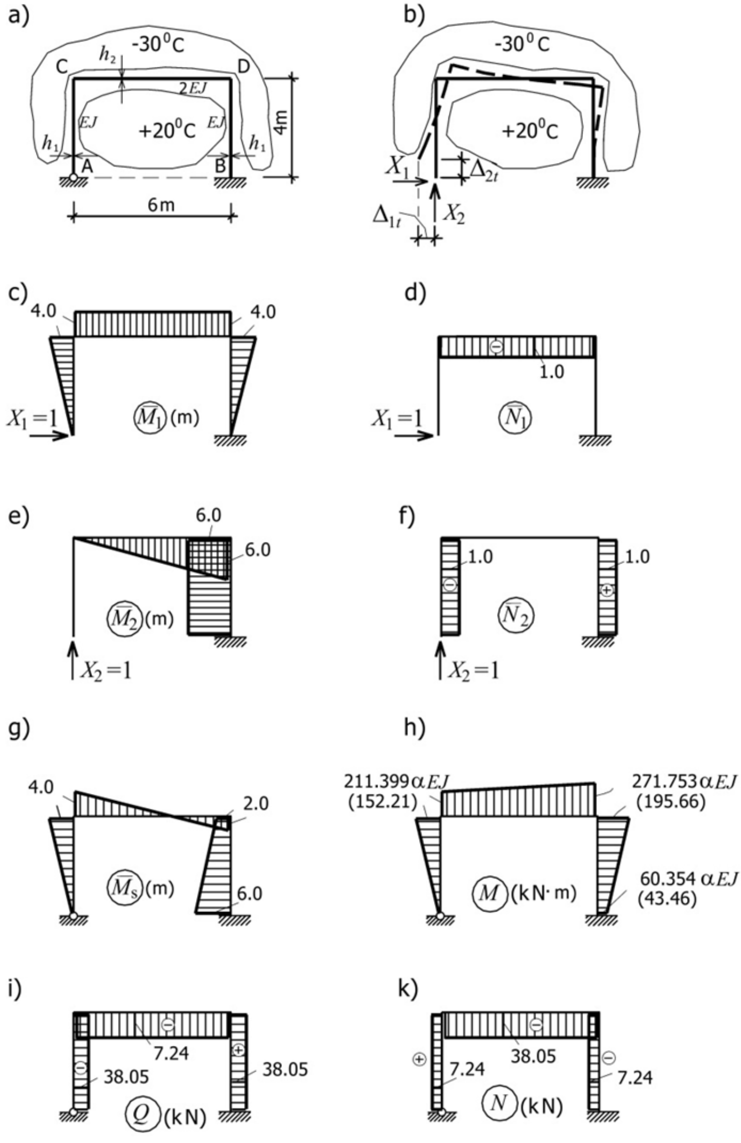
$$\sum\int\frac{M\bar{M}_s\,dx}{EJ}+\sum_{i=1}^{n}\Delta_{it}=0\,. \tag{8.30}$$

When calculating the frames subjected to the settlements of supports, the canonical equations are written in the form:

$$\left.\begin{array}{l}\delta_{11}X_1+\delta_{12}X_2+\delta_{13}X_3+\cdots+\delta_{1n}X_n+\Delta_{1c}=0;\\ \delta_{21}X_1+\delta_{22}X_2+\delta_{23}X_3+\cdots+\delta_{2n}X_n+\Delta_{2c}=0;\\ \cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot\\ \delta_{n1}X_1+\delta_{n2}X_2+\delta_{n3}X_3+\cdots+\delta_{nn}X_n+\Delta_{nc}=0.\end{array}\right\}$$

The free terms of the equations are calculated, in the general case, by the formula (7.13).

E x a m p l e. Construct diagrams M, Q and N caused by the action of temperature change in the frame (Figure 8.21, a). The height of the cross-section of the elements AC and BD equals to $h_1=0.3$ m, the element CD equals to $h_2=0.4$ m. The coefficient of thermal linear expansion of the material equals to $\alpha = 1.2 \cdot 10^{-5}$ 1/(°C), the bendimg rigidity is $EJ = 60$ MN·m^2.



Figures 8.21

The primary system in the initial and deformed states is shown in Figure 8.21, b. The coefficients of the canonical equations will be determined taking into account the influence of only bending moments. Using the diagrams $\bar{M}_1$ and $\bar{M}_2$ (Figures 8.21, e, g), we obtain:

$$\delta_{11} = \frac{272}{3EJ}, \quad \delta_{22} = \frac{180}{EJ}, \quad \delta_{12} = -\frac{84}{EJ}.$$

For the calculating convenience of free terms Δ_{1t} and Δ_{2t} (the corresponding segments are shown in Figure 8.21, b) using formula (7.12), we write the used values of the calculating parameters in the Table 8.1.

Table 8.1

№ element	h, m	t, (°C)	t', (°C)	ω_{M_1}, m^2	ω_{N_1}, m	ω_{M_2}, m^2	ω_{N_2}, m
AC	0.3	–5	50	8	0	0	4
CD	0.4	–5	50	24	6	18	0
BD	0.3	–5	50	8	0	24	4

Recall that in the calculations by formula (7.12) each term in it is assumed to be positive in the case when the corresponding directions of the elements deformation caused by unit forces and thermal action coincide.

$$\Delta_{1t} = \sum \alpha\, t\, \Omega_{N_1} + \sum \frac{\alpha\, t'}{h} \Omega_{M_1} =$$

$$= \alpha \cdot 5 \cdot 6 - \frac{\alpha \cdot 50}{0.3} 8 - \frac{\alpha \cdot 50}{0.4} 24 - \frac{\alpha \cdot 50}{0.3} 8 = -5636.67\, \alpha.$$

$$\Delta_{2t} = \sum \alpha\, t\, \Omega_{N_2} + \sum \frac{\alpha\, t'}{h} \Omega_{M_2} =$$

$$= \alpha \cdot 5 \cdot 4 - \alpha \cdot 5 \cdot 4 + \frac{\alpha \cdot 50}{0.4} 18 + \frac{\alpha \cdot 50}{0.3} 24 = 6250\, \alpha.$$

Following the calculation algorithm (section 8.7), we write the system of canonical equations in numerical form:

$$\left.\begin{aligned}\frac{272}{3EJ}X_1-\frac{84}{EJ}X_2-5636.67\,\alpha=0;\\ -\frac{84}{EJ}X_1+\frac{180}{EJ}X_2+6250.0\,\alpha=0.\end{aligned}\right\}$$

Solving it, we find $X_1=52.8498\,\alpha\,EJ$ kN, $X_2=-10.0590\,\alpha\,EJ$ kN.

The static indeterminacy of the frame disclosed. There is not the diagram of moments caused by the external exposure in the statically determinate primary system subjected to the thermal effect.

Therefore, we construct the final diagram of bending moments by the expression:

$$M=\bar{M}_1X_1+\bar{M}_2X_2\,.$$

This diagram is shown in Figure 8.21, h. Bending moments in the frame depend on the values of the rigidity of the elements, i. e., one of the general properties of statically indeterminate systems is confirmed (Section 8.1). In parentheses are the ordinates for the initial data accepted in the example.

We are performing a kinematic verification. The total diagram of unit moments M_s is shown in Figure 8.21, g.

$$\sum\int\frac{M\ \bar{M}_S}{EJ}+\sum_{i=1}^{2}\Delta_{it}=\frac{\alpha\,EJ}{EJ}\left[\frac{1}{2}211.399\cdot 4\cdot\frac{2}{3}\cdot 4+\right.$$
$$+\frac{6}{6\cdot 2}\left(2\cdot 211.399\cdot 4-2\cdot 271.753\cdot 2+271.753\cdot 4-211.399\cdot 2\right)-$$
$$-\frac{4}{6}\left(2\cdot 271.753\cdot 2+2\cdot 60.354\cdot 6+60.354\cdot 2+271.753\cdot 6\right)-$$
$$\left.-5636.67\,\alpha+6250.0\,\alpha\right]=0.$$

The condition (8.30) is satisfied. The diagrams Q and N are shown in Figures 8.21, i, k.

E x a m p l e. Construct diagrams M, Q and N in the frame subjected to the settlements of supports indicated in Figure 8.22, a. It is assumed that the rigidity of the frame elements equal to $EJ = 60$ MN·m^2, and the settlements of supports equals to $c_1 = c_2 = c = 0.01$ m.

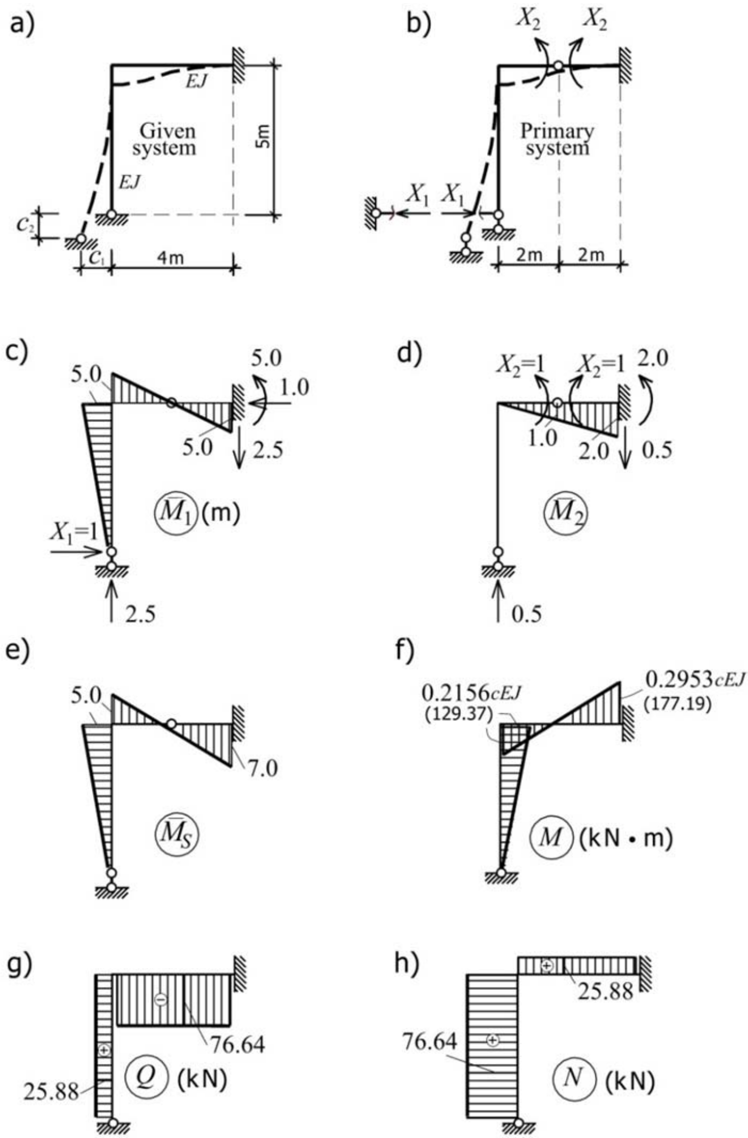


Figure 8.22

The given frame is twice statically indeterminate. Selecting the primary system of the force method (Figure 8.22, b), we write the canonical equations in the form:

$$\left.\begin{aligned}\delta_{11}X_1+\delta_{12}X_2+\Delta_{1c}=0;\\ \delta_{21}X_1+\delta_{22}X_2+\Delta_{2c}=0.\end{aligned}\right\}$$

We will construct the unit moments diagrams (Figures 8.22, c, d) and calculate the coefficients of canonical equations:

$$\delta_{11}=\frac{225}{3EJ},\qquad \delta_{22}=\frac{16}{3EJ},\qquad \delta_{12}=\frac{20}{3EJ}.$$

Considering the distribution of reactions in the support constraints due to $X_1=1$ (Figure 8.22, c) and $X_2=1$ (Figure 8.22, d), according to the formula (7.13) we get:

$$\Delta_{1c}=-\sum R_{k1}c_k=-\left(-1\,c_1-2.5\,c_2\right)=c_1+2.5\,c_2=3.5\,c;$$

$$\Delta_{2c}=-\sum R_{k2}c_k=-\left(-0.5\,c_2\right)=0.5\,c_2=0.5\,c.$$

The canonical equations, after simple transformations, get the following form:

$$\left.\begin{aligned}\frac{225}{3}X_1+\frac{20}{3}X_2+3.5\,cEJ=0;\\ \frac{20}{3}X_1+\frac{16}{3}X_2+0.5\,cEJ=0.\end{aligned}\right\}$$

Having solved them, we find:

$$X_1=-0.043125\,cEJ,\qquad X_2=-0.039844\,cEJ.$$

Since the displacements of the supports does not cause efforts in a statically determinate system the final diagram of the bending moments is constructed by the expression:

$$M=\bar{M}_1X_1+\bar{M}_2X_2.$$

It is shown in Figure 8.22, f. In parentheses there are the values of the ordinates of the moments for the accepted source data. Kinematic verification, as when calculating the thermal effect, is reduced to verifying the fulfillment of the condition:

$$\sum\int \frac{M\bar{M}_s\,dx}{EJ} + \sum\Delta_{ic} = 0\,.$$

We will check it using the total unit diagram M_s (Figure 8.22, e):

$$\frac{cEJ}{EJ}\left[-\frac{1}{2}0.2156\cdot 5\,\frac{2}{3}\,5+\frac{4}{6}\times\right.$$
$$\left.\times\left(-2\cdot 0.2156\cdot 5-2\cdot 0.2953\cdot 7+0.2156\cdot 7+0.2953\cdot 5\right)\right]+$$
$$+3.5\,c+0.5\,c=0.$$

The check is performed. The diagrams Q and N are shown in Figures 8.22, g, h.

8.11. Influence Line for Efforts

To construct the influence line for any effort, it is necessary, first, using the well-known methods of structural mechanics, to obtain the dependence (analytical or numerical) of this effort due to the position of the force $F=1$ $\left(S=f\left(x\right)\right)$, and then, using this dependence, determine the ordinates of the influence line for all characteristic sections.

If the methods of statics are used to determine the dependence $S=f\left(x\right)$ then the corresponding method of constructing the influence line is called static one.

In statically indeterminate systems, the effort in the cross-section of the element is determined by the expression (8.14). If it is used to construct influence lines mind, that the values of the primary unknowns X_i and the value of the effort in the cross-section k of the primary system change due to the moving load $F=1$. Therefore, the expression (8.14) for constructing the influence line for the effort in the cross-section k should be rewritten in the form:

$$\text{inf.line } S_k = \text{inf.line } S_k^0 + \overline{S}_{k1}(\text{ inf.line } X_1) +$$
$$+ \overline{S}_{k2}(\text{ inf.line } X_2) + \ldots + \overline{S}_{kn} \text{ (inf.line } X_n), \tag{8.31}$$

where inf.line S_k^0 is the influence line for effort S in the cross-section k of the primary system;

$\overline{S}_{ki}$ is the effort in the cross-section k of the primary system caused by $X_i = 1$ $(i = 1, 2, \cdots, n)$.

We use this expression to construct the influence line for bending moment in the cross-section k of a once statically indeterminate beam (Figure 8.23, a).

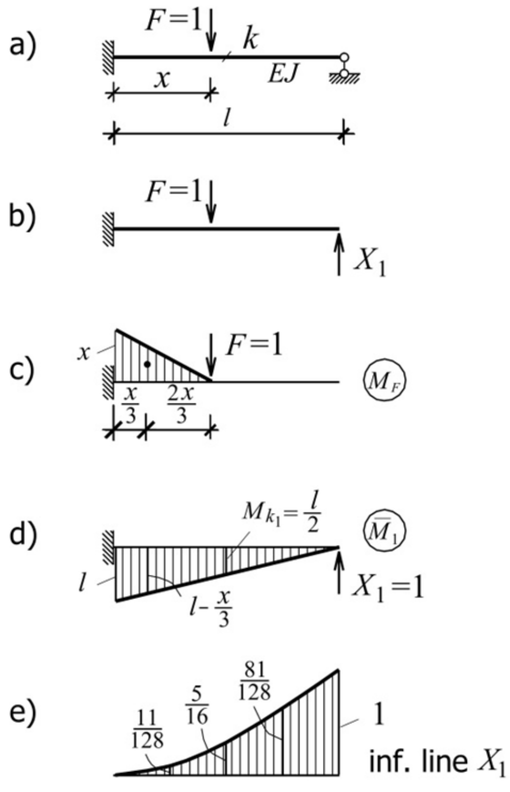


Figure 8.23

Having selected the primary system (Figure 8.23, b), we will construct the diagram of the moments caused by the movable load (Figure 8.23, c) and caused by the unit unknowm $X_1 = 1$ (Figure 8.23, d). Then we determine:

$$\delta_{11} = \frac{l^3}{3EJ}, \qquad \delta_{1F} = -\frac{x^2(3l - x)}{6EJ}.$$

From the canonical equation

$$\delta_{11}X_1 + \delta_{1F} = 0$$

we find:

$$X_1 = \frac{x^2(3l - x)}{2l^3}.$$

It follows that the influence line for X_1 is described by a curve of the third degree relative to the abscissa x of the moving load $F = 1$. It is shown in Figure 8.23, e.

In statically determinate systems, influence lines for efforts have a rectilinear outline or piece-broken (a rectilinear outline on a limited length of the movement of force). Recall, for example, influence lines for support reactions in simple beams, influence lines for bending moments, etc.

To construct influence line for bending moment M_k, the expression (8.31) can be written in the form:

$$\text{inf.line } M_k = \text{inf.line } M_k^0 + \bar{M}_{k1}(\text{inf.line } X_1). \tag{8.32}$$

In this example $M_{k1} = \frac{l}{2}$ (Figure 8.23, d). Inf.line M_k^0 is shown in Figure 8.24, b, and inf.line $M_{k1}\ X_1$ is shown in Figure 8.24, c.

Summing up two last influence lines, we get inf.line M_k (Figure 8.24, d).

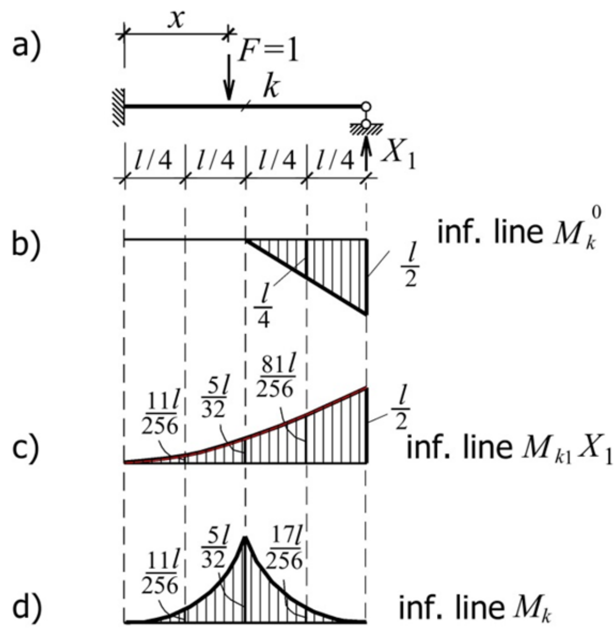


Figure 8.24

The described method of constructing influence line can be applied to systems with a small number of unknowns using the "manual" (non-automated) method of calculating ordinates.

For complex systems, including frames, it is difficult to obtain analytical dependences of the required factor on the coordinate of the load $F=1$, therefore, numerical methods of solution are used for them. Using computer programs that implement methods for calculating various systems, one can find the required effort caused by unit load in various characteristic cross-sections of the frame.

Thus, in order to construct the influence line for an effort, it is necessary to calculate the given system sequentially for several loadings by forces $F=1$ applaied in several characteristic points. Let us explain this approach to constructing of influence lines using the example of a two-span frame (Figure 8.25) all of whose elements have the same ben ding rigidity.

Suppose that a force $F=1$ can move along elements 4–8 and 9–13. We construct influence line for bending moment in cross-section 6.

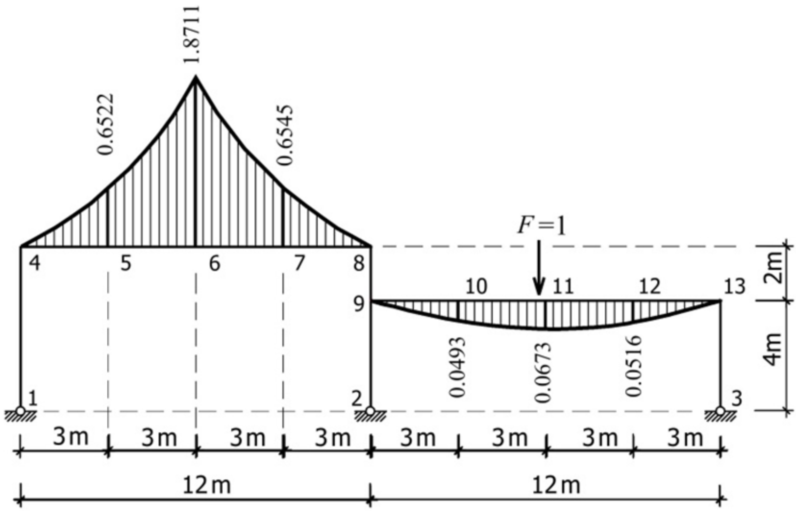


Figure 8.25

We accept three intermediate cross-sections on each of the bars and assume that all the elements of the frame have longitudinal rigidity $EA \to \infty$. Next, we perform the calculation of given frame subjected to the six unit loadings (force $F = 1$ is applied in each intermediate cross-section of elements 4–8 and 9–13). From the calculation results for each load position, we select the bending moment values in cross-section 6 and build with they the influence line M_6 (Figure 8.25).

The static method in the presented form is currently the main method for constructing of influence lines for efforts and displacements in bar and continuum systems.

Such an approach to constructing influence lines for efforts (or other factors) is described in more detail in Section 9.11.

Let us briefly explain the essence of the kinematic method of constructing influence lines for efforts in statically indeterminate frames.

If for the given system having n redundant constraints, we take a statically indeterminate system with $n-1$ redundant constraints as the primary system, then the canonical equation of the force method for calculating the frame for the action of the force $F = 1$ will take the form:

$$\delta_{11}^{(n-1)} X_1 + \delta_{1F}^{(n-1)} = 0 . \tag{8.33}$$

Since by the theorem on reciprocity of displacements $\delta_{1F}^{(n-1)} = \delta_{F1}^{(n-1)}$, then:

$$X_1 = -\frac{\delta_{F1}^{(n-1)}}{\delta_{11}^{(n-1)}}, \tag{8.34}$$

where $\delta_{11}^{(n-1)}$ is the displacement (in the system with $n-1$ unknowns) of the application point of force X_1 in its direction; it is calculated by "multiplying" the diagram $M_1^{(n-1)}$ by itself;

$\delta_{F1}^{(n-1)}$ is the displacement (in the same system) of application point of force $F = 1$, caused by force $X_1 = 1$.

The load $F = 1$ can take any position on the frame elements, therefore, δ_{F1} determines the displacement of the frame elements from the force $X_1 = 1$.

Thus, the expression (8.34) for constructing the influence line for X_1 can be written as follows:

$$\text{inf.line}\, X_1 = -\frac{diag.\, \delta_{F1}^{(n-1)}}{\delta_{11}{}^{(n-1)}}. \tag{8.35}$$

So, to construct an influence line for X_1 it is necessary to construct the displacements diagram caused by the load $X_1 = 1$ of the frame elements along which the force $F = 1$ moves, and divide all its ordinates by $(-\delta_{11})$.

The outline of the influence line turns out to be similar to the displacements diagram of the frame elements. The multiplier $(-\frac{1}{\delta_{11}})$ is the similarity coefficient. This is the main advantage of the kinematic method. With its help it is easy to imagine the shape of the influence line for effort. For this, it is necessary to remove the constraint in which the required force arises and load the frame (or other system) by the appropriate force $X_1 = 1$. With sufficient engineering intuition, it is easy to draw a diagram of displacements, i. e. the shape of influence line.

To construct, for example, the influence line for M_k in the statically indeterminate beam (Figure 8.24, a), it is need to set a hinge in the cross-section k and load the beam with bending moments X_1 (Figure 8.26). The diagram of the vertical displacements of the beam points will be similar to inf. line M_k.

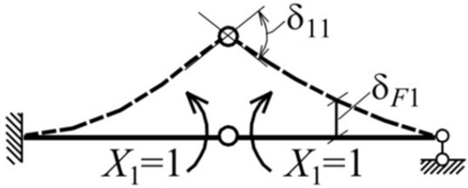


Figure 8.26

To construct the influence line M_6 in the frame (Figure 8.25) we set the hinge in the 6-th cross-section and load the frame with bending moments $X_1 = 1$ (Figure 8.27). The diagram of the vertical displacements of the horisontal elements caused by the given unit moments will be similar to the influence line M_6. The ordinates δ_{F1} of the displacement diagram, if necessary, can be calculated according to the rules set out in section 8.9.

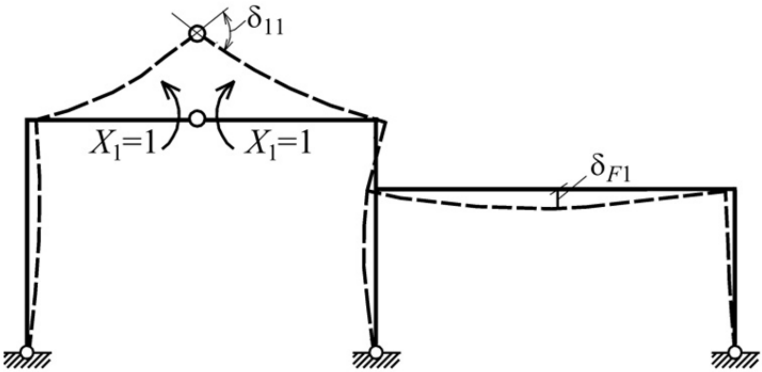


Figure 8.27

THEME 9. DISPLACEMENT METHOD AND ITS APPLICATION TO PLANE FRAMES CALCULATION

9.1. Degre of kinematic indeterminacy. Primary Unknowns

The positions of the ends of the bar, which is part of a loaded frame or other structural system, fully characterize the bar deformed state. Moreover, if the bar adjoins the node rigidly, then the position of its end for a plane system is determined by three parameters: the angle of rotation of the end section and two components of linear displacement, if the connection is articulated, then only by two components of linear displacement. In bar systems, the corresponding displacements of the ends of several bars connected in one node are equal to each other, however, as a rule, they are unknown. Therefore, systems containing such nodes are called kinematically indeterminable. The total number of unknown node displacements is called the degree of kinematic indeterminacy of the system. For example, the frame shown in Figure 9.1 is four times kinematically indeterminable: the movement of node 2 is characterized by three components (Z_1, Z_2, Z_3), node 3 – by one component (Z_4).

Kinematically indeterminable systems are not only statically indeterminate. These, in the general case, include statically determinable systems.

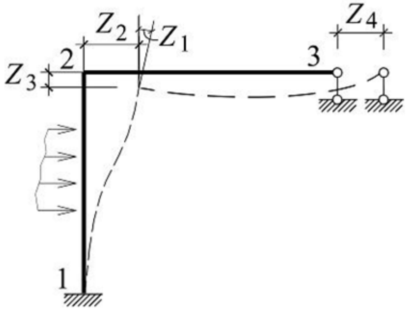


Figure 9.1

For example, the console beam, fixed at one end (Figure 9.2), can be considered as twice kinematically indeterminate if we suppose that at the other end there is a node without links. In turn the statically determinate frame (Figure 9.3) is four times kinematically indeterminate.

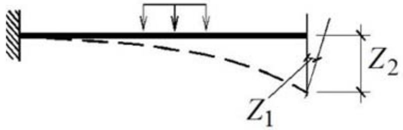


Figure 9.2

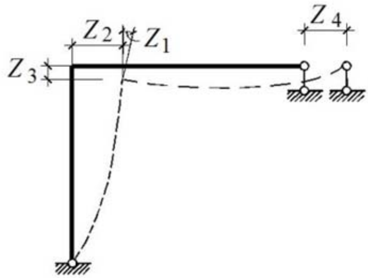


Figure 9.3

If by any method it were possible to find the displacements of the terminal sections of the bar, then the subsequent task of determining the internal forces in its cross sections could be solved quite simply, since for linearly deformable systems there are single-valued relationships between the internal forces, displacements of the nodes and the load.

It is with the help of the displacement method, the essence of which will be described below, that the displacements of the nodes are determined first as the primary unknowns of this calculation method. And only after that the internal forces in the bars are determined. On the contrary, in the force method internal forces were determined first, and then displacements.

Earlier, in the first chapter, the main assumptions for a linearly deformable system were indicated. In addition to them, when calculating frames by the displacement method, the following assumptions are introduced:

1. The deformations of the bars caused by transverse forces are not taken into account.

2. The influence of longitudinal deformations is alsow not taken into account (calculation of frames taking into account longitudinal deformations will be considered in Section 9.12).

3. The initial length of the straight bar is assumed to be equal to the length of the chord, tightening its ends after deformation.

These assumptions can significantly reduce the number of primary unknowns of displacement method. So, for the frame (Figure 9.1), the movement of node 2 can already be characterized by only two components: the angle of rotation Z_1 and horizontal displacement Z_2. Since the longitudinal deformation of the bar 1–2 is not taken into account, then the vertical displacement $Z_3 = 0$. In a deformed state, the position of the node 2 should be shown on line 2–3 (the arc described by the radius $r = l_{1-2}$ from center 1 is replaced by a tangent to it at point 2). The horizontal displacement of node 3, due to the third assumption, must be taken equal Z_2. Therefore, the frame in question is twice kinematically indeterminate. The deformed frame diagram and primary unknowns, modified in accordance with the accepted assumptions, are shown in Figure 9.4, a.

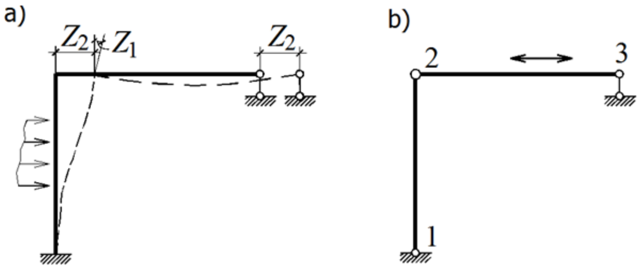


Figure 9.4

From the above reasoning for the image of the deformed state of the frame, it follows that the total number n of primary unknowns of the displacement method is determined as the sum of the unknown rotation angles n_a of the rigid nodes and independent linear displacements n_l of the all nodes, i.e.

$$n = n_a + n_l.$$

Number n_a is the degree of angular mobility, and n_l – the degree of linear mobility of the nodes. Moreover, if the definition of n_a is reduced to counting the number of rigid nodes, then to determine the degree of linear mobility n_l, the given frame must be turned into a hinged-rod system by introducing hinges into all rigid nodes, including supporting ones, and determine the degree of freedom W for it. When turning the frame into a hinge-rod system, the statically determinate consoles can be discarded (the degree of linear mobility decreases).

For the frame under consideration (Figure 9.4, a) $n_a = 1$, and for the corresponding hinge-rod system (Figure 9.4, b) $W = 1$, that is $n_l = 1$. Kinematic analysis of the design scheme shows that nodes 2 and 3 possess mobility in the horizontal direction (it is shown by the arrows $\longleftrightarrow$). Therefore, $n = n_a + n_l = 1 + 1 = 2$.

For another frame (Figure 9.5, a) the degree of linear mobility of its nodes will be found using the hinge-rod system (Figure 9.5, b), for which $W = 3$.

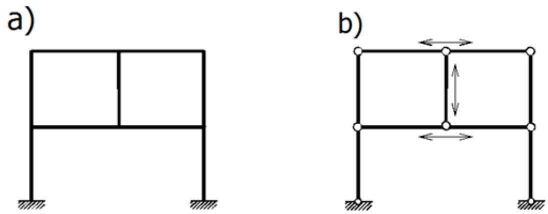


Figure 9.5

The independent directions of the nodes motions are shown in this figure by arrows. The total number of primary unknown displacements is $n = n_a + n_l = 6 + 3 = 9$.

9.2. Primary system

Calculation of the frame by the method of displacements for a given loading will begin with the fact that we first accept the unknown displacements of the nodes equal to zero. We fix this state of the frame, i.e. we fix all the nodes with unknown displacements using additional links that prevent the angular and linear movements of the nodes. Obviously, the number of additional angular links will be equal n_a, and the number of linear ones – n_l.

The links that prevent angular movements are the so-called floating rigid supports. They do not allow rigid nodes to rotate, while the linear mobility of the nodes is not limited. In such links, the only kind of reactions possible is the moment. On the design diagrams of the frames, they are represented by shaded squares. Linear links must be set in the directions of possible independent linear displacements of nodes. The system thus obtained is called the primary system of the displacement method.

As an example, the Figure 9.6, a shows the primary system of the displacement method for the frame shown in Figure 9.4, a. The Figure 9.6, b contains the primary system for the frame shown in Figure 9.5, a.

The nodes displacements of the primary system are known: they are equal to zero. Therefore, the primary system can be called kinematically determinate.

An analysis of the structural interaction of the primary system elements (Figure 9.6) shows that the primary system of the displacement method consists of single-span independent beams with hinged and/or absolutely rigid supports at the ends.

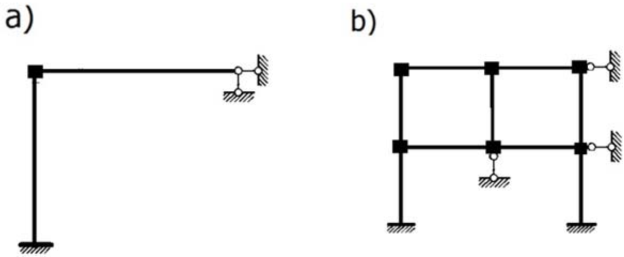


Figure 9.6

When transferring any of the beams that make up the primary system into a state corresponding to its deformed position in a given frame under load (Figure 9.7), internal forces arise in it. In accordance with the principle of independence of the forces action (principle of superposition), these efforts can be represented as the sum of the efforts caused by:

1. The action of the load located on the beam (bar).
2. By turning the left end of the bar (and the right end if the bar is pinched at both ends) by angle Z_A equal to the true value of the angle of rotation of node A.
3. Mutual linear displacement Δ_{AB} of the ends of the bar in a direction perpendicular to its axis.

Supporting reactions arising in a statically indeterminable beam of constant stiffness under various influences on it serve as auxiliary quantities in the calculation of frames by the displacement method. Their values can be found by the method of forces.

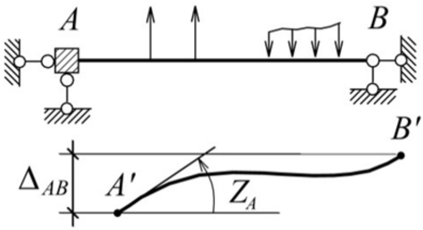


Figure 9.7

E x a m p l e 1. Let us define the reactions in the support links and plot the bending moments diagram in the beam loaded with a uniformly distributed load (Figure 9.8, a). Having accepted the primary system of the force method in the form of a cantilever beam (Figure 9.8, b), we construct a unitare (Figure 9.8, c) and load (Figure 9.8, d) diagram of bending moments.

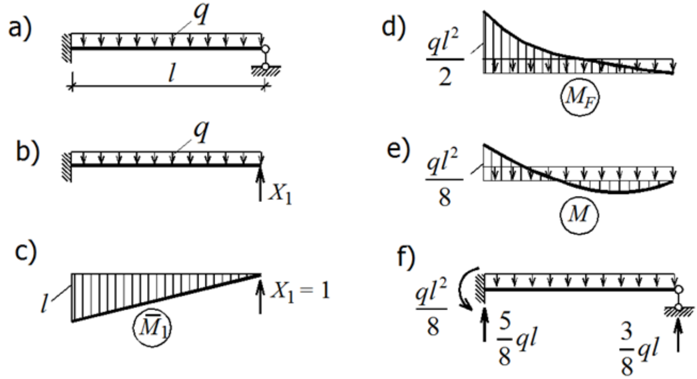


Figure 9.8

The canonical equation of the force method has the form

$$\delta_{11}X_1 + \Delta_{1F} = 0,$$

where

$$\delta_{11} = \frac{l^3}{3EJ}, \quad \Delta_{1F} = -\frac{1}{EJ}\frac{1}{3}\frac{ql^2}{2}l\frac{3}{4}l = -\frac{ql^4}{8EJ}.$$

Therefore,

$$X_1 = -\frac{\Delta_{1F}}{\delta_{11}} = \frac{3}{8}ql\,.$$

The final diagram of bending moments, constructed by expression $M = M_F + \overline{M}_1 X_1$, is shown in Figure 9.8, d, and in Figure 9.8, f the values of the support reactions are shown.

E x a m p l e 2. Consider a beam loaded with a concentrated force F (Figure 9.9, a). We construct the bending moments diagram M_F using the primary system from Example 1 (Figure 9.9, b).

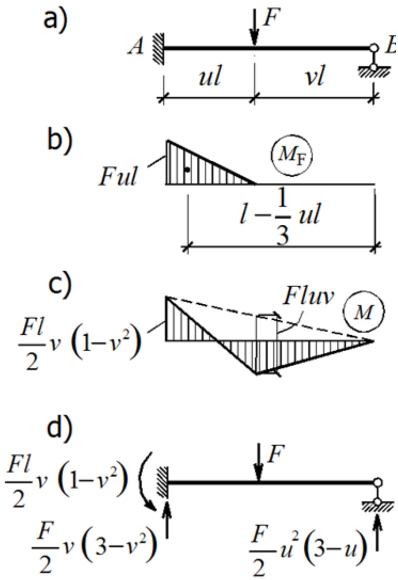


Figure 9.9

We calculate the free term of the canonical equation

$$\Delta_{1F} = -\frac{1}{EJ}\,\frac{1}{2}\,F\,ul\,ul\left(l - \frac{1}{3}ul\right) = -\frac{F}{6EJ}u^2 l^3 \left(3-u\right).$$

As $\delta_{11} = \dfrac{l^3}{3EJ}$, then $X_1 = \dfrac{F}{2}u^2\left(3-u\right)$.

From the equilibrium condition $\sum Y = 0$ it follows that

$$R_A = F - X_1 = \frac{F}{2} v \left(3 - v^2\right).$$

The final diagram of bending moments is shown in Figure 9.9, c. The values of the support reactions are given in Fugure. 9.9, d.

E x a m p l e 3. Let us plot the bending moments diagram from the rotation of the clamped end of the beam at angle φ (Figure 9.10, a).

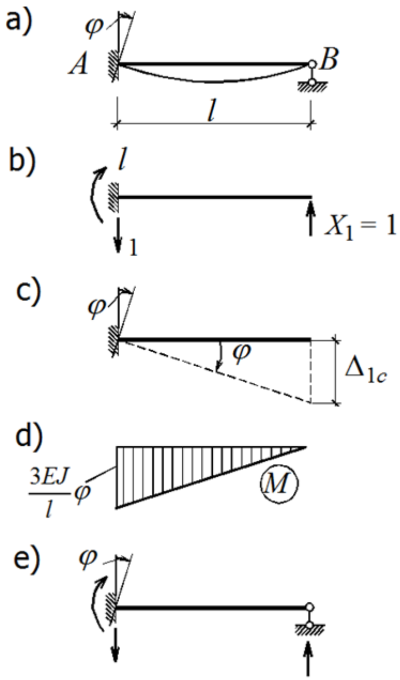


Figure 9.10

We write the canonical equation of the force method in the form

$$\delta_{11} X_1 + \Delta_{1c} = 0.$$

The free term can be calculated by the expression

$$\Delta_{1c} = -\sum R_{k1} c_k ,$$

where R_{k1} is the reactions in the link "k" of the primary system caused by force $X_1 = 1$ (Figure 9.10, b).

$$\Delta_{1c} = -(l\,\varphi) = -l\,\varphi.$$

The same value Δ_{1c} can be obtained from the kinematic analysis of the disign scheme (Figure 9.10, c): displacement of point B is opposite to the direction of force X_1.

Then

$$X_1 = -\frac{\Delta_{1c}}{\delta_{11}} = \frac{3EJ}{l^2}\varphi.$$

The bending moments diagram and the distribution of support reactions are shown in Figure 9.10, d and e.

E x a m p l e 4. Let us determine the efforts in the beam from the displacement of the fixed support by amount Δ in the direction perpendicular to beam axis (Figure 9.11, a).

As in example 3, the displacement Δ_{1c} can be found, using the formula:

$$\Delta_{1c} = -\sum R_{k1} c_k = -(-1 \cdot \Delta) = \Delta .$$

The support reaction at point B, equal to X_1, can be found as

$$X_1 = -\frac{\Delta_{1c}}{\delta_{11}} = -\frac{3EJ}{l^3}\Delta .$$

Diagram M and the values of the support reactions are shown in Figures 9.11, b, c.

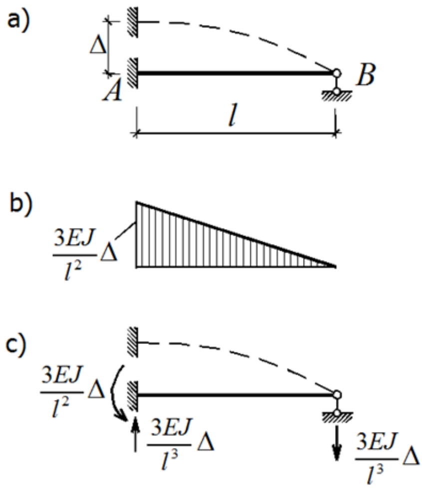


Figure 9.11

E x a m p l e 5. As an external influence on the beam, we consider thermal one (Figure 9.12, a).

The canonical equation of the force method for calculating at the temperature change is

$$\delta_{11}X_1 + \Delta_{1t} = 0.$$

Assuming $t_1 > t_2$, we will depict the deformed state of the primary system in Figure 9.12, b.

Value Δ_{1t} is found by the formula:

$$\Delta_{1t} = \frac{\alpha t'}{h}\Omega_M = -\frac{\alpha t'}{h}\frac{l^2}{2},$$

where $t' = t_1 - t_2$.

The solution of the canonical equation gives

$$X_1 = \frac{3EJ\,\alpha\,t'}{2h\,l}.$$

The bending moments diagram is shown in Figure 9.12, c and supports reactions are shown in Figure 9.12, d.

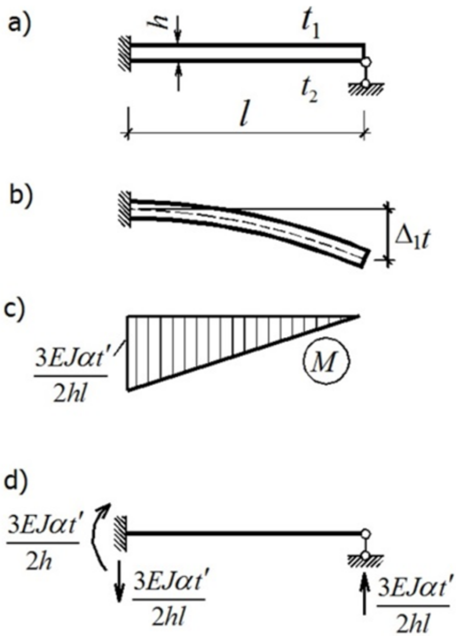


Figure 9.12

E x a m p l e 6. We show the calculation of a twice statically indeterminate beam at rotation of fixed support A by angle φ (Figure 9.13, a). The primary system of the force method can be chosen as symmetric one (Figure. 9.13, b). The corresponding unit diagrams of bending moments are presented in Figures 9.13, c and d. The state of the primary system caused by the rotation of fixed support A at angle φ is shown in Figure 9.13, e. Since $\delta_{12} = \delta_{21} = 0$, the canonical equations for determining the primary unknowns are represented in the form:

$$\delta_{11} X_1 + \Delta_{1c} = 0,$$

$$\delta_{22} X_2 + \Delta_{2c} = 0.$$

The coefficients at the unknown are:

$$\delta_{11} = \frac{l^3}{12EJ}; \qquad \delta_{22} = \frac{l}{6EJ}.$$

The free terms of the equations are:

$$\Delta_{1c} = -\sum R_{k1} c_k = -\left(-\frac{l}{2}\varphi\right) = \frac{l}{2}\varphi,$$

$$\Delta_{2c} = -\sum R_{k2} c_k = -(1 \cdot \varphi) = -\varphi.$$

Solving the equations gives

$$X_1 = -\frac{6EJ}{l^2}\varphi, \qquad X_2 = \frac{EJ}{l}\varphi.$$

The bending moments diagram in the beam is represented in Figure 9.13, f. The support reactions are shown in Figure 9.13, g.

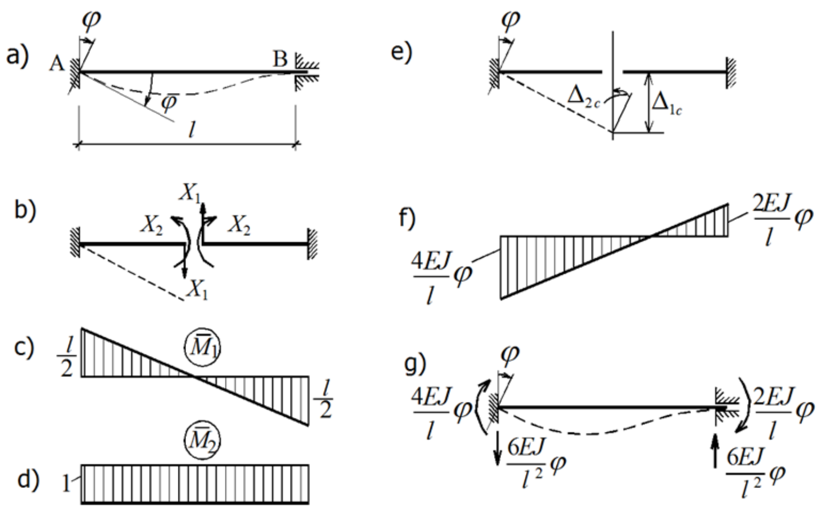


Figure 9.13

The results of calculations of such beams for various types of loads are given in Table 9.1. This table will be used at calculating frames with the displacement method.

Table 9.1

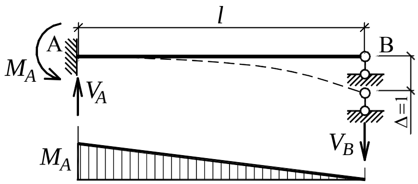
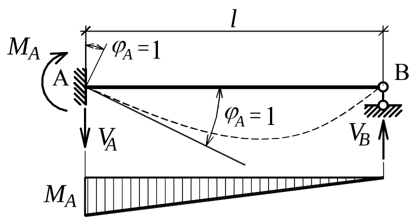
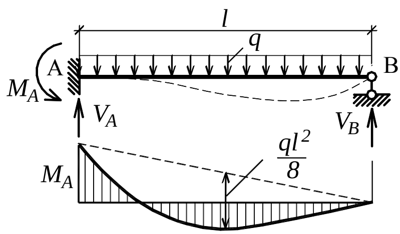
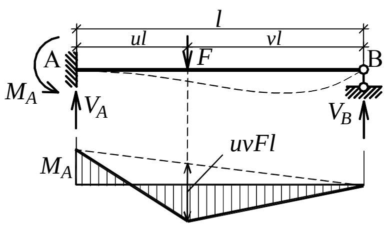
№№	Beams schemes and bending moments diagrams	Formulas for determining reactions
1	2	3
1		$i=\dfrac{EJ}{l}$; $M_A=\dfrac{3i}{l}$; $V_A=V_B=\dfrac{3i}{l^2}$
2		$M_A=3i$; $V_A=V_B=\dfrac{3i}{l}$
3		$M_A=\dfrac{ql^2}{8}$; $V_A=\dfrac{5}{8}ql$; $V_B=\dfrac{3}{8}ql$
4		$M_A=\dfrac{Fl}{2}\nu\left(1-\nu^2\right)$; $V_A=\dfrac{F\nu}{2}\left(3-\nu^2\right)$; $V_B=\dfrac{Fu^2}{2}(3-u)$

Table 9.1 (continuation)

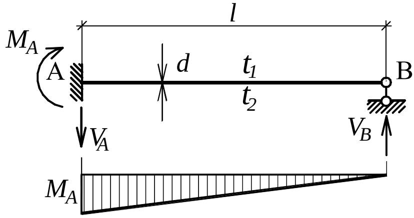
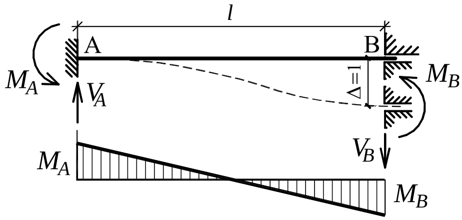
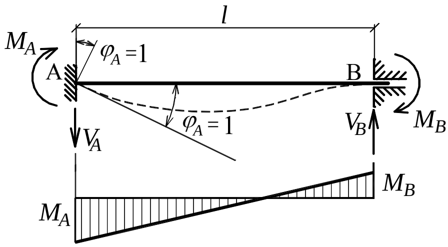
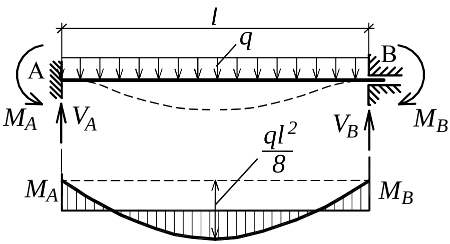
1	2	3
5	Uneven heating  	$M_A = \dfrac{3EJ\alpha t'}{2d}$; $V_A = V_B = \dfrac{3EJ\alpha t'}{2dl}$; α – linear expansion coefficient; $t_1 > t_2$; $t' = t_1 - t_2$
6	 	$M_A = M_B = \dfrac{6i}{l}$; $V_A = V_B = \dfrac{12i}{l^2}$
7	 	$M_A = 4i$; $M_B = 2i$; $V_A = V_B = \dfrac{6i}{l}$
8	 	$M_A = M_B = \dfrac{ql^2}{12}$; $V_A = V_B = \dfrac{ql}{2}$

Table 9.1 (continuation)

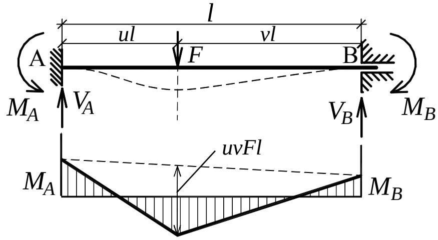
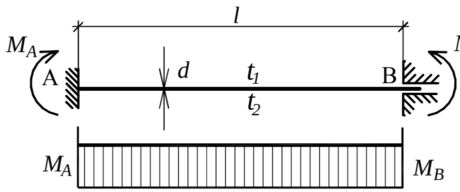
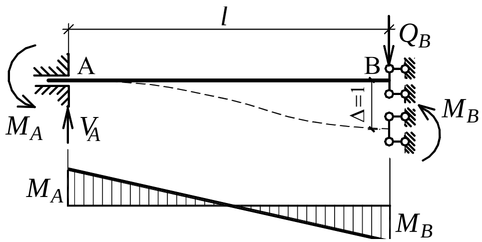
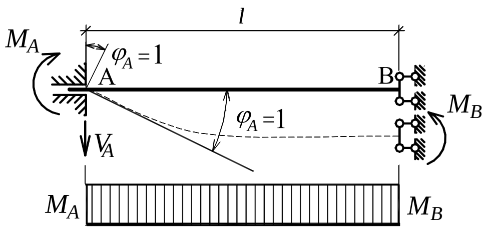
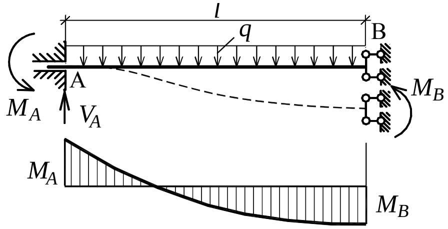
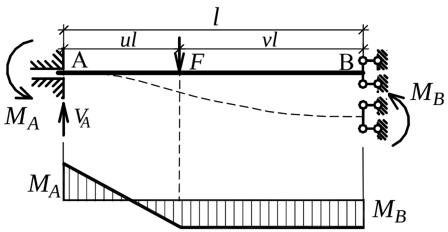
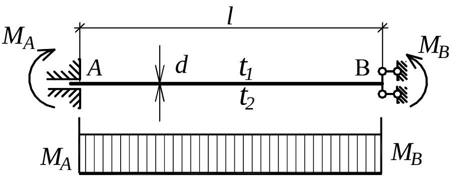
1	2	3
9		$M_A = uv^2 Fl$; $M_B = u^2 v Fl$; $V_A = v^2\left(1+2u\right)F$; $V_B = u^2\left(1+2v\right)F$
10	Uneven heating 	$M_A = M_B = \dfrac{i\alpha t' l}{d}$; $V_A = V_B = 0$; α – linear expansion coefficient; $t_1 > t_2$; $t' = t_1 - t_2$
11		$M_A = M_B = \dfrac{6i}{l}$; $V_A = Q_B = \dfrac{12i}{l^2}$
12		$M_A = M_B = i$; $V_A = V_B = 0$

Table 9.1 (ending)

1	2	3
13		$M_A = \frac{ql^2}{3}$; $M_B = \frac{ql^2}{6}$; $V_A = ql$; $V_B = 0$
14		$M_A = \frac{Fl}{2} u\left(2-u\right)$; $M_B = \frac{Fl}{2} u^2$; $V_A = F$; $V_B = 0$
15		$M_A = M_B = \frac{i\alpha t'l}{d}$; $V_A = V_B = 0$; α – linear expansion coefficient; $t_1 > t_2$; $t' = t_1 - t_2$

9.3. Canonical Equations

The forces in the elements of the primary system change when it is transfered into a position corresponding to the deformed position of the given system. From the given effect reactions arise in additional links of the primary system. If displacements equal to the displacements in the corresponding directions of the given system are given each additional angular and linear link, then the reactions in the additional links must be equal to zero.

Consequently, reactions in the additional links are functions of nodal displacements and loads, and the condition of static equivalence of the primary and given systems is reduced to equations of the form

$$R_i\left(Z_1, Z_2, \cdots, Z_n, F\right) = 0,\ i = \overline{1, n}, \qquad (9.1)$$

where R_i is the complete reaction in the i-th additional link caused by displacements and load.

The number of such equations is naturally equal to the total number of displacement method unknowns.

Based on the principle of independence of the action of forces, functional dependence (9.1) can be represented as

$$R_i = R_{i1} + R_{i2} + \cdots + R_{in} + R_{iF} = 0, \qquad (9.2)$$

where R_{ik} is the reaction in link i caused by the true value of the displacement of link k $\left(k = \overline{1, n}\right)$; R_{iF} is the reaction due to load.

Value R_{ik} can be write in the form

$$R_{ik} = r_{ik}\ Z_k, \qquad (9.3)$$

where r_{ik} is the reaction in link i caused by the unit value of the displacement of link k ($Z_k = 1$); Z_k is a true offset value in the direction of link k.

Substituting (9.3) into equation (9.2) and accepting $i = 1, 2, \cdots, n,$ we obtain the following system of linear equations:

$$\left.\begin{array}{llllll} r_{11}Z_1 + & r_{12}Z_2 + & \cdots + & r_{1n}Z_n + & R_{1F} & = 0, \\ r_{21}Z_1 + & r_{22}Z_2 + & \cdots + & r_{2n}Z_n + & R_{2F} & = 0, \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{n1}Z_1 + & r_{n2}Z_2 + & \cdots + & r_{nn}Z_n + & R_{nF} & = 0. \end{array}\right\} \qquad (9.4)$$

These equations are called the canonical equations of the displacement method. As follows from the previous reasoning, the physical meaning of i-th equation is that the total reaction in this additional link caused by displacements $Z_1, Z_2, \cdots, Z_n$ and a given external load has to be zero.

Coefficients (reactions) $r_{11}, r_{22}, \cdots, r_{nn}$ located on the main diagonal are called the main reactions; coefficients (reactions) r_{ik} $(i \neq k)$ are called secondary, and free terms $R_{1F}, R_{2F}, \cdots, R_{nF}$ are called load reactions.

When the structure is exposed to temperature changes, the free terms of the equations are replaced by $R_{1t}, R_{2t}, \cdots, R_{nt}$, and when there are support shifts – by $R_{1c}, R_{2c}, \cdots, R_{nc}$.

When determining the reaction in i-th additional link, its positive direction should coincide with the positive direction of displacement Z_i adopted in the primary system.

In the matrix notation, the equations (9.4) have the form:

$$R\,Z + R_F = 0\,, \tag{9.5}$$

where

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ . & . & . & . \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}$$

is matrix of coefficients of canonical equations (system rigidity matrix in the directions of additional links);

$$Z = \left[Z_1, Z_2, \cdots, Z_n\right]^T$$

is matrix (vector for one load option) of primary unknowns;

$$R_F = \left[R_{1F}, R_{2F}, \cdots, R_{nF}\right]^T$$

is matrix (vector for one load option) of the free members of the canonical equations (load reactions).

9.4. Static method for determining the coefficients and free terms of canonical equations

Coefficients and free terms of canonical equations are reactions in additional links. To determine them, it is necessary to know the distribution of forces in the primary system due to the unit displacements of these links and due to the load.

Plotting diagrams of bending moments from these effects are shown on the example of a two-span frame (Figure 9.14, a).

The same figure also shows the possible deformed state of the frame, which allows you to visually determine the number of primary unknowns: one angular and the other linear. We confirm, however, determining the number of unknowns according to the general rules. The frame has one rigid node. As follows from the kinematic analysis of the hinge-rod system (Figure 9.14, b), the degree of linear mobility of its nodes is also equal to one. The possible direction of movement of the nodes in the figure is shown by the arrow $\leftrightarrow$. The total number of unknowns is equal $n = n_a + n_l = 1 + 1 = 2$. The primary system is shown in Figure 9.14, c.

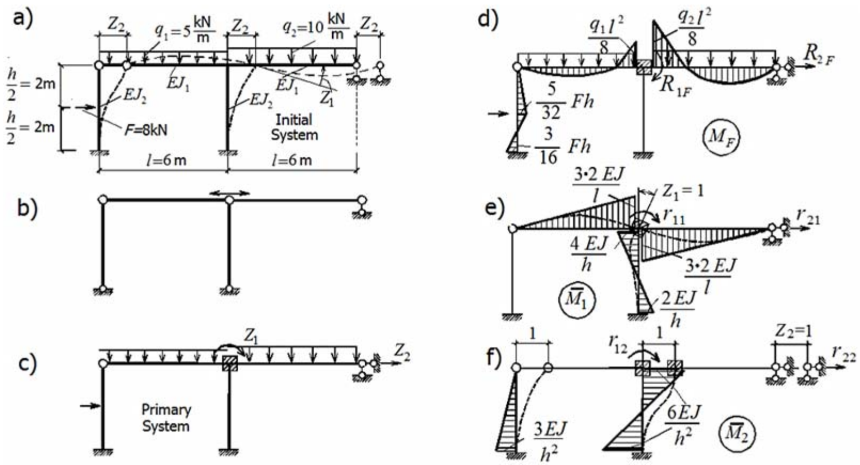


Figure 9.14

Plotting the bending moment diagrams is performed using the data in table 9.1. Diagram M_F (load diagram) and diagrams $\bar{M}_1$, $\bar{M}_2$ (unit diagrams) are shown in Figures 9.14, d, e, f.

In addition, in Figures 9.14, e, f the dashed line shows the curved axes of the bars, which allow you to set the position of the stretched fibers on each of them and correctly depict the plot of the moments. The same figures show the reactions in the additional links. Their directions are accepted as positive (in the directions of the positive displacement of the links). Recall that in the designation of the reaction r_{ik}, the first index (i) indicates the number of the link in which the reaction occurs, and the second (k) indicates the number of the displacement that caused this reaction. In the designation R_{iF}, the second index (F) means that the cause of the reaction is load F.

To determine the reactions by a static way equilibrium equations are used. In particular, since only a moment can occur in a floating support, to determine it, one should use the equation of equilibrium of the form $\sum M = 0$. So, to determine R_{1F}, we will show the internal forces acting on the node in the cut-out state (Figure 9.15, a), and draw up the equation:

$$R_{1F} + \frac{q_2 l^2}{8} - \frac{q_1 l^2}{8} = 0,$$

from which we find

$$R_{1F} = \frac{q_1 l^2}{8} - \frac{q_2 l^2}{8}.$$

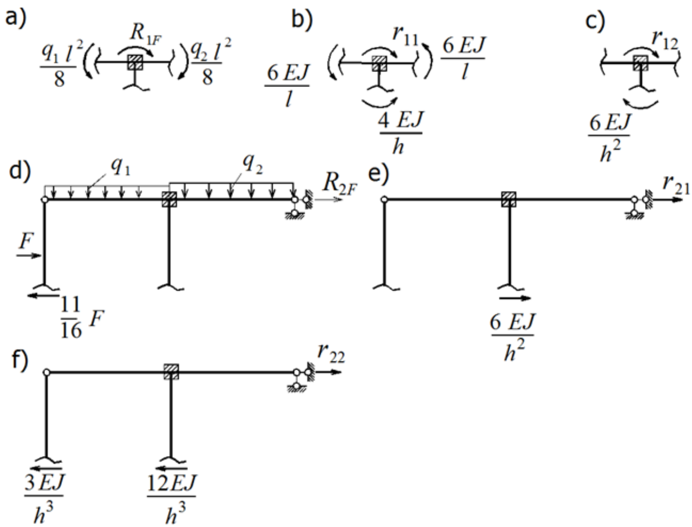


Figure 9.15

The equilibrium equation for determination of the unit reaction r_{11} (Figure 9.15, b) can be written as follows:

$$r_{11} - \frac{6EJ}{l} - \frac{6EJ}{l} - \frac{4EJ}{h} = 0.$$

Therefore,

$$r_{11} = \frac{12EJ}{l} + \frac{4EJ}{h}.$$

An equation also in the form of the sum of moments relative to the node (Figure 9.15, c)

$$\sum M = r_{12} + \frac{6EJ}{h^2} = 0$$

is used to determine the unit reactive moment r_{12}:

$$r_{12} = -\frac{6EJ}{h^2}.$$

Reactions in additional links that impede linear movements of nodes are determined from the equilibrium conditions of a frame fragment. All external and internal forces acting on the fragment, except the reaction to be calculated, must be known.

For the example under consideration, when determining the reactions R_{2F}, r_{21}, r_{22}, such fragments can be the frame diagrams shown in Figures 9.15, d, e, f.

Writing down the corresponding conditions of equilibrium of forces shown in each of these figures, we obtain equations for determining unknown reactions

In particular, the load reaction R_{2F} (Figure 9.15, d) is determined by the equation

$$\sum X = 0: \quad R_{2F} + F - \frac{11}{16}F = 0, \quad R_{2F} = -\frac{5}{16}F;$$

the secondary unit reaction r_{21} (Figure 9.15, e) by the equation:

$$\sum X = 0, \qquad r_{21} + \frac{6EJ}{h^2} = 0, \qquad r_{21} = -\frac{6EJ}{h^2};$$

and the main unit reaction r_{22} (Figure 9.15, f) by the equation:

$$\sum X = 0, \qquad r_{22} - \frac{12EJ}{h^3} - \frac{3EJ}{h^3} = 0, \qquad r_{22} = \frac{15EJ}{h^3}.$$

9.5. Kinematic Method for Determining the Coefficients and Free Terms of Canonical Equations

Consider some basic system of the method of displacements in unit states “k” and “m” (Figure 9.16, a, b). The work of the external forces of state “k” on the displacements of state “m” is:

$$W_{km} = r_{mk} \cdot 1.$$

It is known that the work of external forces W_{km} is equal (with a minus sign) to the work of internal forces. Therefore, expressing the work of internal forces through bending moments $\bar{M}_k$ in state "k" on the corresponding deformations $\frac{\bar{M}_m\, dx}{EJ}$ of the frame in state "m", we obtain:

$$r_{mk} = \sum \int \frac{\bar{M}_k \bar{M}_m\, dx}{EJ}. \tag{9.6}$$

Calculation of integrals of the form

$$\int \frac{\bar{M}_k \bar{M}_m\, dx}{EJ}$$

reduces to numerical integration, in the simplest cases – to “multiplication” of the bending moments diagrams.

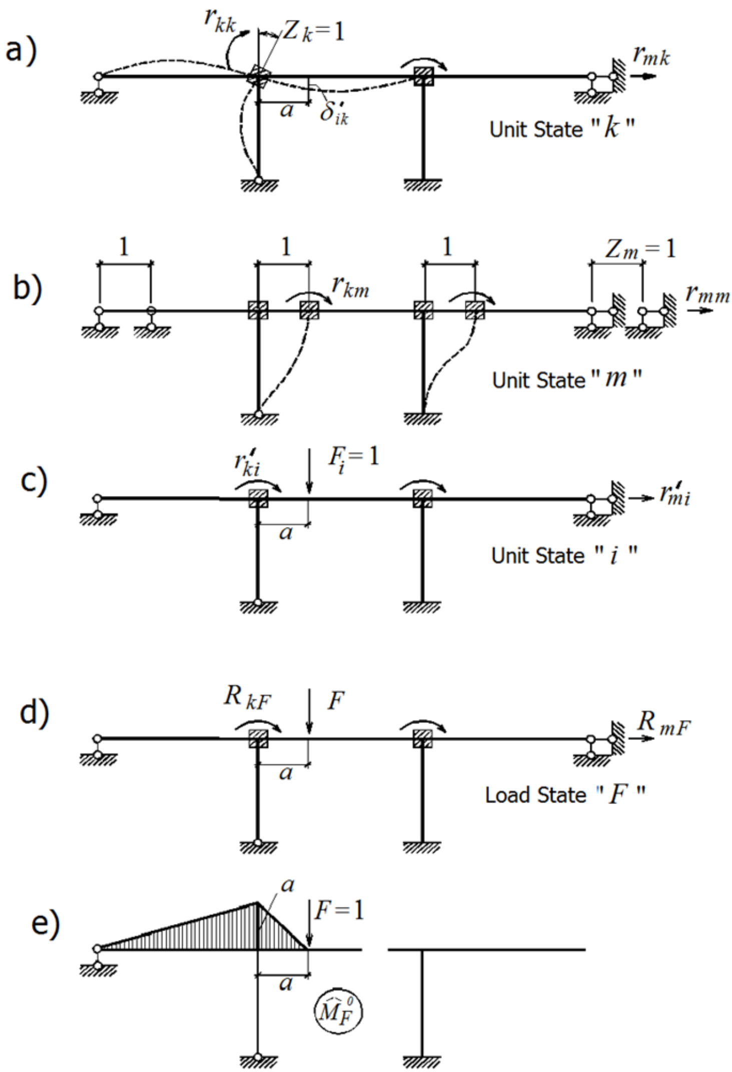


Figure 9.16

Therefore, the coefficients of the canonical equations of the displacement method can be calculated in the same way as the coefficients of the equations of the force method by "multiplying" the corresponding diagrams of bending moments.

Having determined the work of the external forces of state “m” on the displacements of state “k” we get:

$$W_{mk} = r_{km} \cdot 1.$$

Based on the reciprocity theorem, we get:

$$W_{km} = W_{mk},$$

or

$$r_{mk} = r_{km}. \tag{9.7}$$

It is a formal record of the reaction reciprocity theorem (the first theorem of J. Rayleigh (1842–1919)): the reaction in link “m” from the unit displacement of link “k” in its direction is equal to the reaction in the link “k” from the unit displacement of link “k” in its direction.

Consider two more frame states. In the first state "k", unit displacement $Z_k = 1$ is specified (Figure 9.16, a). In the second state “i”, unit force $F_i = 1$ is given (Figure 9.16, c).

Work of the external forces of state “k” on the displacements of state “i” (there are no movements of the nodes) is zero:

$$W_{ki} = 0\,.$$

Therefore, the work of internal forces is equal to zero too:

$$\int \frac{\bar{M}_k \bar{M}_i \, dx}{EJ} = 0\,.$$

However, the work of the external forces of state “i” on the displacements of state “k” is:

$$W_{ik} = 1 \cdot \delta'_{ik} + r'_{ki} \cdot 1\,.$$

By the reciprocity theorem we obtain:

$$W_{ik} = W_{ki} = 1 \cdot \delta'_{ik} + r'_{ki} \cdot 1 = 0.$$

Consequently,

$$\delta'_{ik} = -r'_{ki} . \tag{9.8}$$

Expression (9.8) is a formal notation of the theorem on reciprocity of reactions and displacements (second theorem of J. Rayleigh): the displacement of the point of application of force $F_i = 1$ in its direction, caused by the unit displacement of link "k", is numericly equal (with the opposite sign) to the reaction in link "k" due to unit force $F_i = 1$.

The dimensions of reactions and displacements in this expression are the same. They are installed like this:

Dimension $r'_{ki} =$	$\dfrac{\text{Dimension of the reaction in link "}k\text{"}}{\text{Dimension of force "}F\text{"}}$
Dimension $\delta'_{ik} =$	$\dfrac{\text{Dimension of the displacement in the direction of force "}F\text{"}}{\text{Dimension of the displacement in the direction of link "}k\text{"}}$

To determine the free terms, we consider the primary system of the displacement method in state "k" and in state "F" (load state, Figure 9.16, d).

On the basis of the reciprocity theorem, there is equality:

$$W_{kF} = W_{Fk} .$$

Revealing this equality, we obtain:

$$R_{kF} \cdot 1 + F \cdot \delta'_{ik} = 0 .$$

From the last equality it follows

$$R_{kF} = -F\ \delta'_{ik} .$$

To determine the displacement δ'_{ik} (Figure 9.16, a) in a statically indeterminate system using the Mohr formula, as you know, one of two "multiplied" diagrams of bending moments can be constructed in a statically determinate system obtained from the given system. In this case, it is the primary system of the displacement method.

Then, denoting the diagram of bending moments due to $F = 1$ in a statically determinate system by $\hat{M}_F^0$ (Figure 9.16, e), we obtain

$$\delta'_{ik} = \sum\int \frac{\bar{M}_k \hat{M}_F^0 \, dx}{EJ} . \qquad (9.9)$$

If the external load is a group of forces, then $\hat{M}_F^0$ is a bending moments diagram constructed in the primary system of the force method due to a generalized unit force corresponding to the nature of a given exposure.

Substituting the value of δ'_{ik} into the expression for R_{kF}, we obtain:

$$R_{kF} = -F\sum\int \frac{\bar{M}_k \hat{M}_F^0 \, dx}{EJ} .$$

By introducing F to the integrand and denoting

$$M_F^0 = F \, \hat{M}_F^0 ,$$

we find:

$$R_{kF} = -\sum\int \frac{\bar{M}_k M_F^0 \, dx}{EJ} . \qquad (9.10)$$

So, the calculation of the load reaction R_{kF} is reduced to the calculation of the expression (9.10), in which:

$\bar{M}_k$ is a unit diagram of bending moments constructed in the primary system of the displacement method;

M_F^0 is a diagram of bending moments from the given load, built in a statically determinate system, obtained either from the primary system of the displacement method with the mandatory rejection of link "*k*", or obtained from the given statically indeterminate system, i.e. constructed in the primary system of the force method .

E x a m p l e. Let us determine by kinematic method the reactions r_{12}, R_{1F} and R_{2F} for the frame shown in Figure 9.14.

$$r_{12} = \sum\int \frac{\bar{M}_1 \bar{M}_2 \, dx}{EJ} = \frac{h}{6EJ}\left[-\frac{4EJ}{h}\frac{6EJ}{h^2} - \frac{2EJ}{h}\frac{6EJ}{h^2}\right] = -\frac{6EJ}{h^2}.$$

To determine R_{1F}, we construct in a statically determinate system obtained from the primary system of the method of displacements (Figure 9.14, c) a diagram of the bending moments $M_F^{0(a)}$ (Figure 9.17, a). The index (a) in the designation $M_F^{0(a)}$ corresponds to a variant of the primary system (Figure 9.17, a):

$$R_{1F} = -\sum\int \frac{\bar{M}_1 M_F^{0(a)} \, dx}{EJ} =$$

$$= -\left(-\frac{1}{2EJ}\frac{2}{3}\frac{q_1 l^2}{8} l \frac{3\cdot 2EJ}{2l} + \frac{1}{2EJ}\frac{2}{3}\frac{q_2 l^2}{8} l \frac{3\cdot 2EJ}{2l}\right) = \frac{q_1 l^2}{8} - \frac{q_2 l^2}{8}.$$

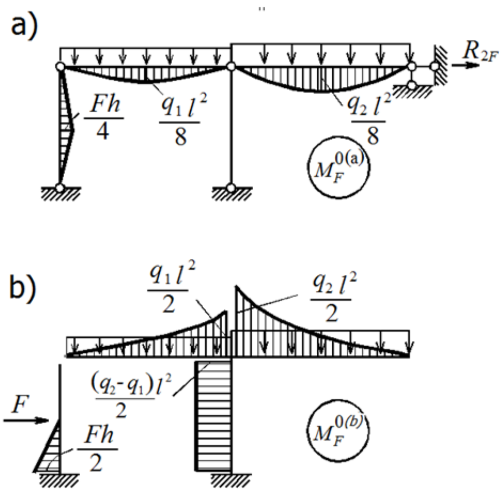


Figure 9.17

To determine R_{2F}, we select a statically determinate system obtained from the given system (Figure 9.14, a), and construct a diagram of the $M_F^{0(b)}$ (Figure 9.17, b):

$$R_{2F} = -\sum\int \frac{\bar{M}_2 M_F^{0(b)} dx}{EJ} = -\left(\frac{1}{EJ}\frac{1}{2}\frac{Fh}{4}\frac{h}{2}\frac{5EJ}{2h^2}\right) = -\frac{5}{16}F.$$

9.6. Building and Checking Internal Forces Diagrams Due to External Loads

Having solved the system of canonical equations (9.4), we find the values of the primary unknowns of the displacements method. To build the final diagram of bending moments, it is necessary to first build the adjusted unit diagrams $\bar{M}_i Z_i$ (they are called "corrected" unit diagrams of moments). The final diagram M of bending moments from an external load in the given statically indeterminate system is constructed by summing the load diagram M_F with the "corrected" unit diagrams, i. e., the ordinate of the diagram M in each concrete section "k" of the frame bar is calculated by the formula:

$$M_k = M_{kF} + \bar{M}_{k1} Z_1 + \bar{M}_{k2} Z_2 + \cdots + \bar{M}_{kn} Z_n .$$

The main verification of the correctness of the final diagram of bending moments M in the displacement method is a static check, which, as is known, reduces to checking the equilibrium of moments in the frame nodes.

In addition, as in the force method, a kinematic check can be applied to check the correctness of the final diagram M: the result of "multiplying" each unit moments diagram (or total unit diagram) of the force method by the final moments diagram should be zero

The diagram of transverse forces Q is constructed, as in the method of forces, according to diagram M, and the diagram of the longitudinal forces N is constructed according to diagram Q. Static check of diagrams Q and N is performed in the same way as in the force method (in the displacement method, static check refers to the main one).

E x a m p l e 1. Let us construct a diagram of bending moments in the frame shown in Figure 9.18, a.

To reduce the number of unknowns, when calculating the degree of linear mobility of the frame nodes, the console, as a statically determinate fragment, is discarded. Then the degree of freedom W of the hinge-rod system (Figure 9.18, b) will be equal to unity, that is $n_l = 1$. The total number of the displacement method unknowns is equal $n = n_a + n_l = 2 + 1 = 3$. At Figure 9.18, c the primary system and the positive directions of the primary unknowns are shown, and Figure 9.18, d–g shows the bending moments diagram in the primary system, due to load, and unit diagrams of bending moments.

The system of canonical equations has the form:

$$\left.\begin{aligned} r_{11}Z_1 + \quad r_{12}Z_2 + \quad r_{13}Z_3 + \quad R_{1F} &= \quad 0,\\ r_{21}Z_1 + \quad r_{22}Z_2 + \quad r_{23}Z_3 + \quad R_{2F} &= \quad 0,\\ r_{31}Z_1 + \quad r_{32}Z_2 + \quad r_{33}Z_3 + \quad R_{3F} &= \quad 0. \end{aligned}\right\}$$

We note some features of calculating the coefficients at the unknowns and free terms of the canonical equations. To determine coefficient r_{32}, we write the condition for the equilibrium of the fragment (Figure 9.18, h) of the design scheme taken from Figure 9.18, f:

$$\sum X = 0; \quad r_{32} - 0.24EJ + 0.375EJ = 0; \quad r_{32} = -0.135EJ .$$

Using the data (Figure 9.18, g), we can verify that the reciprocity of the reactions is observed: $r_{23} = r_{32}$. Indeed, from the condition of the equilibrium of moments in the node (Figure 9.18, i) it follows that

$$\sum M = 0; \quad r_{23} - 0.24EJ + 0.375EJ = 0; \quad r_{23} = -0.135EJ.$$

Free term R_{3F} can be determined from the equilibrium equation for the fragment (Figure 9.18, j) obtained from (Figure 9.18, d):

$$\sum X = 0; \quad R_{3F} + 15.0 = 0; \quad R_{3F} = -15.0.$$

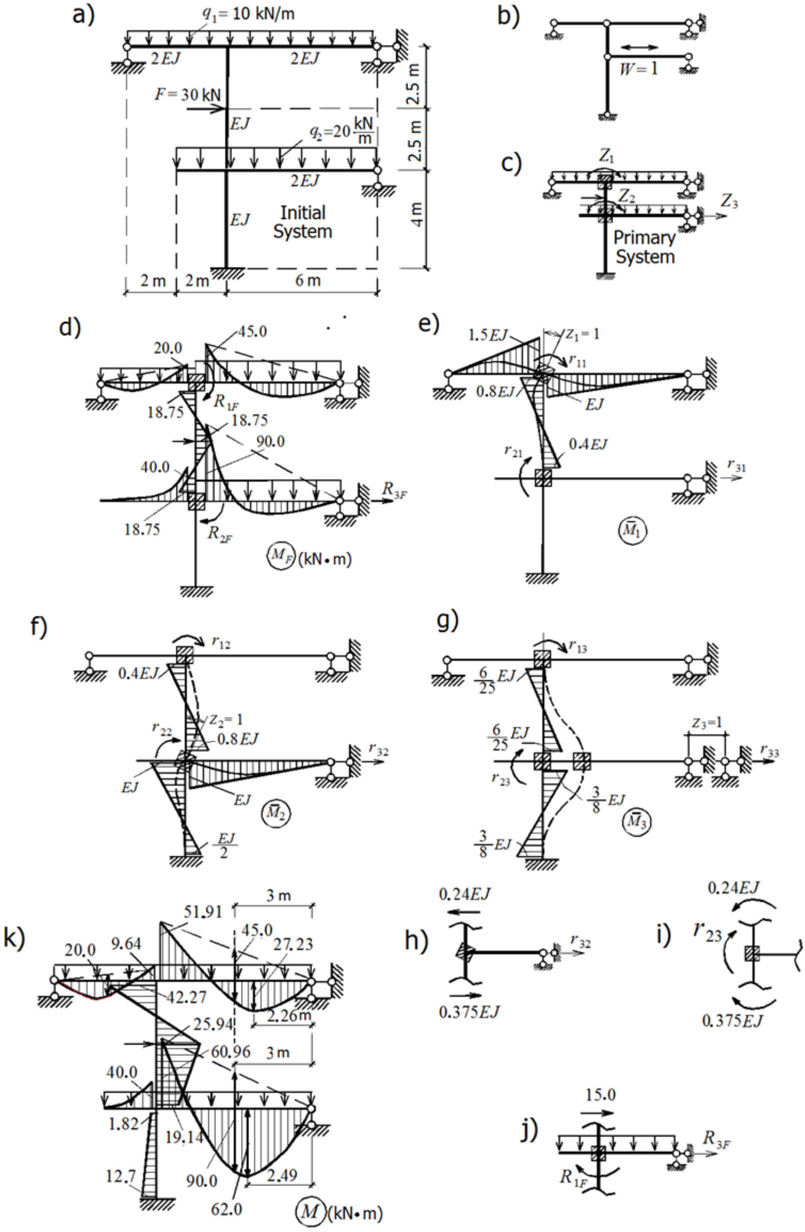


Figure 9.18

In numerical form, the system of canonical equations is written as follows:

$$\left.\begin{aligned}
3.3\,Z_1 + 0.4\,Z_2 + 0.24\,Z_3 - 6.25\frac{1}{EJ} &= 0,\\
0.4\,Z_1 + 2.8\,Z_2 - 0.135\,Z_3 - 68.75\frac{1}{EJ} &= 0,\\
0.24\,Z_1 - 0.135\,Z_2 + 0.2835\,Z_3 - 15.0\frac{1}{EJ} &= 0.
\end{aligned}\right\}$$

Its solution is:

$$Z_1 = -6.905\frac{1}{EJ}\ \text{rad},\quad Z_2 = 29.040\frac{1}{EJ}\ \text{rad},\quad Z_3 = 72.584\frac{1}{EJ}\ \text{m}.$$

The final diagram of bending moments is shown in Figure 9.18, k.

E x a m p l e 2. Let us construct a diagram of bending moments for the frame shown in Figure 9.19, a.

To determine the degree of linear mobility of the nodes, we use a hinged-rod system (Figure 9.19, b). The total number of unknowns is equal $n = n_a + n_l = 2 + 1 = 3$. The primary system of the displacement method and the positive directions of the primary unknowns are shown in Figure 9.19, c. The load and unit diagrams of bending moments are shown in Figure 9.19, d,…,g.

The system of canonical equations in numerical form has the form:

$$\left.\begin{aligned}
2.6EJ\,Z_1 + 0.8EJ\,Z_2 - 0.375EJ\,Z_3 - 38.0 &= 0,\\
0.8EJ\,Z_1 + 3.2EJ\,Z_2 - 15.0 &= 0,\\
-0.375EJ\,Z_1 + 0.234375EJ\,Z_3 - 60.0 &= 0.
\end{aligned}\right\}$$

Its solution is:

$$Z_1 = 72.361\frac{1}{EJ}\ \text{rad},\qquad Z_2 = -13.403\frac{1}{EJ}\ \text{rad},$$

$$Z_3 = 371.778\frac{1}{EJ}\ \text{m}.$$

The final diagram of bending moments is shown in Figure 9.19, h.

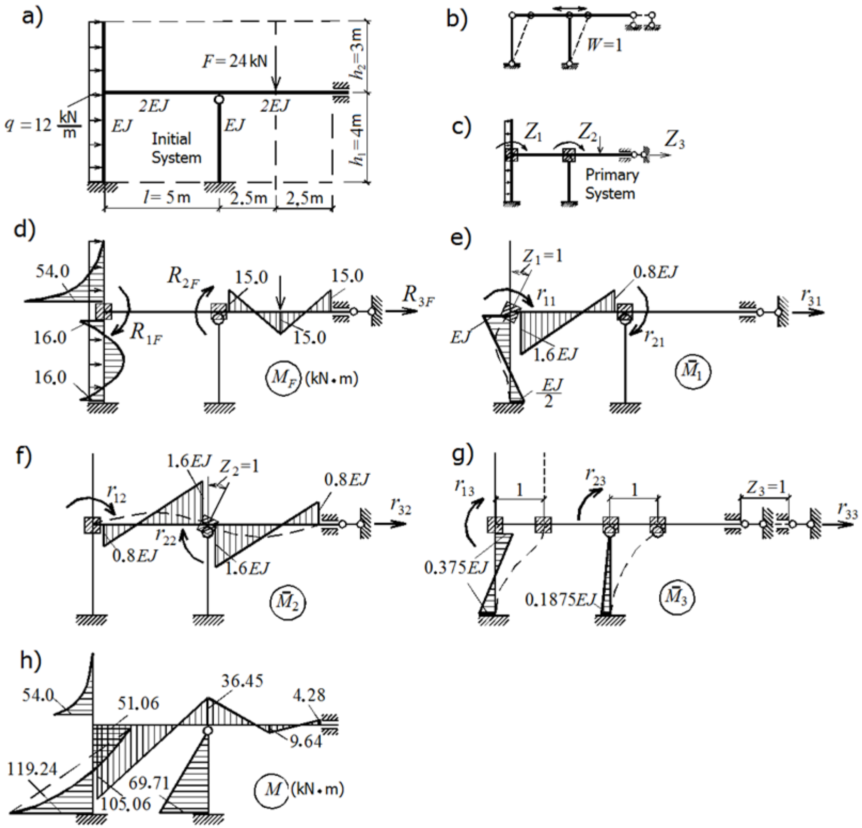


Figure 9.19

9.7. Calculating Frames with Inclined Elements

In frames with inclined elements, the displacement of the linear link of a node to a predetermined value, for example, equal to one causes linear displacements of other nodes, which depend not only on the given displacement, but also on the geometry of the frame (the location of its elements). Therefore, to build diagrams of bending moments in the primary system, it is necessary, first of all, to find the mutual displacements of the ends of the bars forming the frame. The displacement values are found from the displacement analysis of the hinge-rod system corresponding to the given frame.

When building any unit diagram only one additional link is displaced by a distance equal to unity, the rest remain motionless. In this case the hinge-rod system is a kinematic mechanism with one degree of freedom. With a known displacement of one node, the displacements of the others can be determined from the movement diagram of the mechanism. Let us explain this with the following examples.

Let us consider a frame with one inclined column (Figure 9.20, a).

The degree of its kinematic indeterminacy is equal to $n = 2$. The link that impedes the linear movement of nodes we place perpendicular to the rod 2–3 (Figure 9.20, b).

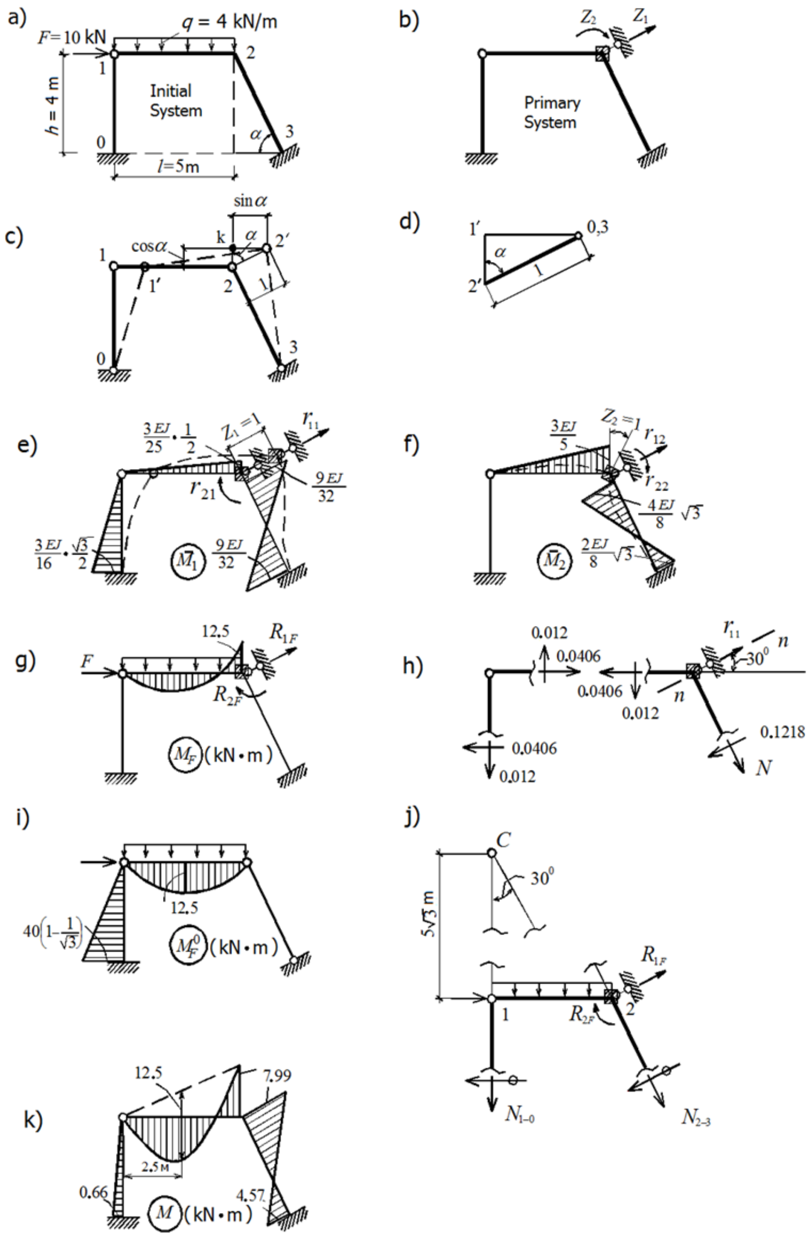
To determine the mutual displacements of the ends of the frame rods at $Z_1 = 1$, we give the hinged-rod system position 0–1′ – 2′ – 3, which is possible under the conditions of its fastening (Figure 9.20, c). The arcs described by points 1 and 2 when the rods rotate around the reference points 0 and 3 are replaced by their tangents in points 1 and 2. From the right triangle 2-k-2′ it follows that the mutual displacement of the ends of rod 1–2 is $\cos\alpha$, of rod 0–1 is $\sin\alpha$, of rod 2–3 is 1.

The same displacement values are obtained from the displacement diagram (Figure 9.20, d). Let us explain its construction.

The support nodes 0 and 3 are fixed. The corresponding point on the displacement diagram is called the pole. From this pole (point 0,3 (Figure 9.20, d)) we draw rays perpendicular to the rods 0–1 and 2–3, i.e., along the directions of possible movements of nodes 1 and 2. On the ray perpendicular to rod 2–3, at a distance equal to unity, point 2′ will lie.

To determine the position of point 1′, it is necessary to draw a line from point 2′ perpendicular to rod 1–2. Segments 1′ – 2′ and 1′ – 0 in the displacement diagram are equal to the mutual displacements of the ends of rods 1–2 and 0–1 for $Z_1 = 1$.

E x a m p l e 1. Let us construct a diagram of bending moments for the frame shown in Figure 9.20, a. The rigidity in bending of all bars are assumed to be the same. Angle $\alpha = \pi / 3$ rad. The length of bar 2–3 is $l_{2-3} = 8/\sqrt{3}$ m.



a)
F=10 kN
q = 4 kN/m
h = 4 m
Initial System
l=5m
b)
Primary System
c)
d)
e)
f)
g)
(kN•m)
h)
i)
j)
k)
(kN•m)

Figure 9.20

To build unit diagram $\bar{M}_1$ (Figure 9.20, e) we use the previously found values of the mutual linear displacements of the ends of the bars. Diagrams $\bar{M}_2$ and M_F are shown in Figures 9.20, f, g.

The coefficients and free terms of the canonical equations are determined by static or kinematic methods. We show, for example, the definition of r_{11} and R_{1F}:

$$r_{11} = \sum\int \frac{\bar{M}_1^2 dx}{EJ} = \frac{1}{EJ}\left(\frac{1}{2} \cdot \frac{3\sqrt{3}\,EJ}{32} \cdot 4 \cdot \frac{2}{3} \cdot \frac{3\sqrt{3}\,EJ}{32} + \right.$$

$$\left. + \frac{1}{2} \cdot \frac{3\,EJ}{50} \cdot 5 \cdot \frac{2}{3} \cdot \frac{3\,EJ}{50} + \frac{1}{2} \cdot \frac{9\,EJ}{32} \cdot \frac{8}{2\sqrt{3}} \cdot \frac{2}{3} \cdot \frac{9EJ}{32} \cdot 2 \right) = 0.1629EJ.$$

The same value can be obtained by the static method from the equilibrium equation of node 2 (Figure 9.20, h). We obtain the transverse and longitudinal forces in rod 1–2 from the equilibrium equation for node 1. The multiplier EJ in the notation in the figure of the transverse and longitudinal forces is omitted.

$$\sum F_{n-n} = 0,$$

$$r_{11} = \left(0.1218 + 0.0406 \cdot \frac{\sqrt{3}}{2} + 0.012 \cdot \frac{1}{2} \right) EJ = 0.1629\,EJ.$$

To calculate R_{1F} by the kinematic method, we use the diagram of bending moments M_F^0 (Figure 9.20, i), built in the primary system of the method of forces:

$$R_{1F} = -\sum\int \frac{\bar{M}_1 M_F^0 dx}{EJ} = -\frac{1}{EJ}\left[-\frac{2}{3} \cdot 12.5 \cdot 5 \cdot \frac{3EJ}{100} + \right.$$

$$\left. + \frac{1}{2} \cdot 40 \left(1 - \frac{1}{\sqrt{3}} \right) \cdot 4 \cdot \frac{2}{3} \cdot \frac{3\sqrt{3}\,EJ}{32} \right] = -2.41.$$

From the equilibrium equation $\sum M_c = 0$ for frame fragment (Figure 9.20, j) we find the same value R_{1F}.

We write the canonical equations in numerical form.

$$\left.\begin{aligned} 0.1629Z_1 - 0.2213Z_2 - 2.41\frac{1}{EJ} = 0, \\ -0.2213Z_1 + 1.4660Z_2 + 12.50\frac{1}{EJ} = 0. \end{aligned}\right\}$$

Their solution gives:

$$Z_1 = 4.0409\frac{1}{EJ}\ \text{m}, \qquad Z_2 = -7.9168\frac{1}{EJ}\ \text{rad}.$$

The final diagram of bending moments (Figure 9.20, k) is obtained according to the formula:

$$M = M_F + M_1 Z_1 + M_2 Z_2.$$

E x a m p l e 2. Let us build diagrams of internal forces for the frame (Figure 9.21, a), assuming that the stiffnesses of all bars on bending are constant and the same. As in the previous examples, we do not take into account the longitudinal deformation of the bars ($EA \to \infty$).

The degree of kinematic indeterminacy of the frame is three. We select the main system (Figure 9.21, b) and build the load diagrams of bending moments M_F (Figure 9.21, e) and unit diagrams of bending moments M_1 and M_2 from angular unknowns $Z_1 = 1$ and $Z_2 = 1$ (Figure 9.21, f, g).

To build a unit diagram from linear displacement $Z_3 = 1$, it is necessary to determine the mutual displacements of the ends of the bars.

First, a given system (Figure 9.21, a) is converted into a hinge-rod system (Figure 9.21, c). Then the hinge-rod system (Figure 9.21, c) receives an offset along the entered additional linear link by $Z_3 = 1$. The nodes 3, 4, 5 are moved to a new position 3', 4', 5'. Next a displacement diagram is constructed (Figure 9.21, d).

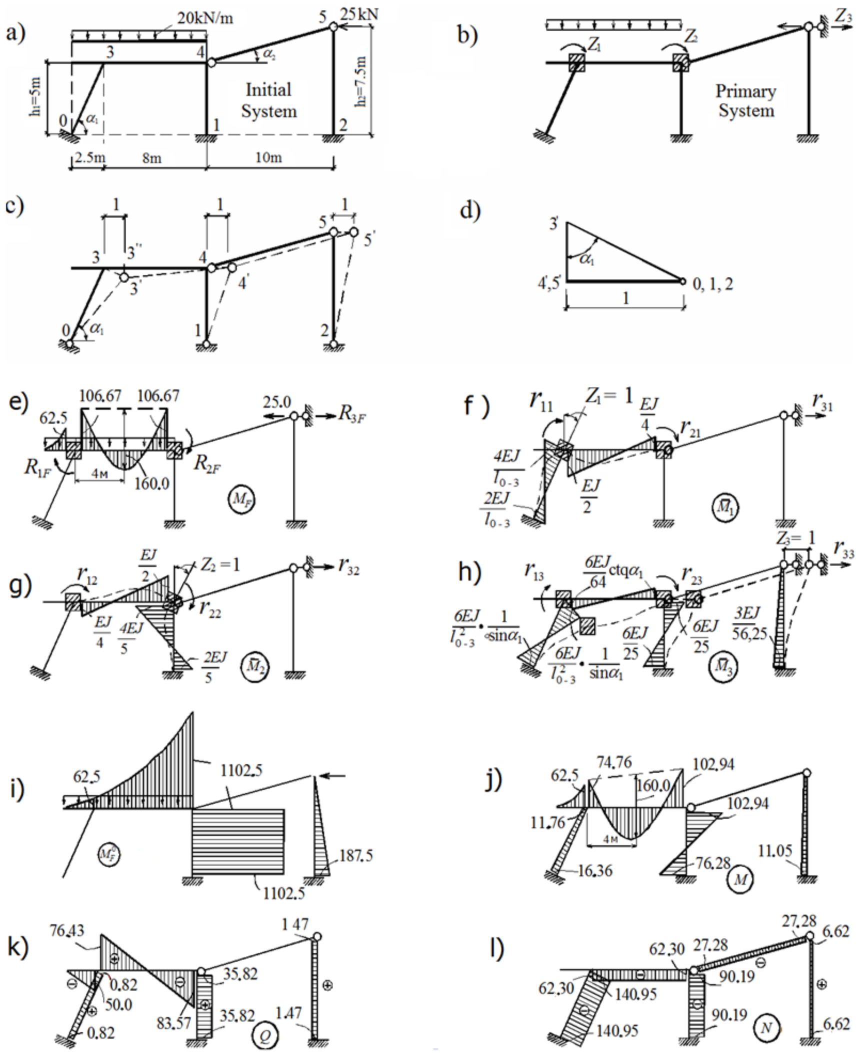


Figure 9.21

On this diagram the lengths of segments 1-4' and 2-5' are equal to one. It is the mutual displacement of the nodes of the rods 1–4 and 2–5 (these vertical rods have different heights, but since they are parallel, the nodes 4 and 5 move horizontally into equal segments). Section 0–3' is equal to the displacement of the node 3 in the direction perpendicular to the rod 0–3; mutual vertical displacement of the ends of the rod 3–4 is determined by the length of the segment 3'–4'.

Unit diagram $\boldsymbol{M}_3$ is presented in Figure 9.21, h.

Coefficient $\boldsymbol{r}_{33}$ of the canonical equation, as well as free term $\boldsymbol{R}_{3\mathrm{F}}$, is conveniently calculated by the kinematic method. One of the possible variants of diagram M_F^0 for determining $\boldsymbol{R}_{3\mathrm{F}}$ is shown in Figure 9.21, i.

After determining the coefficients and free terms, the system of canonical equations is written in the form:

$$\left.\begin{array}{rcrcrcrl}
1.2155\,Z_1 & + & 0.25\,Z_2 & - & 0.167785\,Z_3 & - & 44.17\dfrac{1}{EJ} & =0, \\
0.25\,Z_1 & + & 1.30\,Z_2 & - & 0.193125\,Z_3 & + & 106.67\dfrac{1}{EJ} & =0, \\
-0.167785\,Z_1 & - & 0.193125\,Z_2 & + & 0.194830\,Z_3 & - & 40.0\dfrac{1}{EJ} & =0.
\end{array}\right\}$$

Its solution is:

$$Z_1 = 78.593\frac{1}{EJ}\ \text{rad}, \qquad Z_2 = -66.389\frac{1}{EJ}\ \text{rad},$$

$$Z_3 = 207.182\frac{1}{EJ}\ \text{m}.$$

The final diagrams of the internal forces M, Q and N are shown in Fiures 9.21, j, k, l.

9.8. Using System Symmetry

It is known that any load acting on a symmetric system (Figure 9.22, a) can be represented as the sum of symmetric and inverse-symmetric components.

The first of them corresponds to the symmetric form of deformation of the frame (Figure 9.22, b), and the second to the inverse-symmetric (Figure 9.22, c). Therefore, the rotation angles of the nodes of the given frame (Figure 9.22, a) can be found as the sum or difference of symmetric and inverse-symmetric unknowns:

$$\varphi_1 = Z_1 + Z_2, \qquad \varphi_2 = Z_1 - Z_2.$$

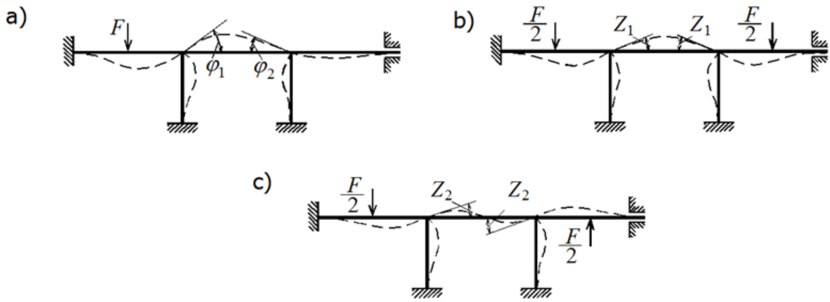


Figure 9.22

Similar relations hold for linear displacements. For example, for a frame (Figure 9.23, a) it is possible to calculate its linear displacements Δ1 and Δ2, using symmetric (Figure 9.23, b) and inverse-symmetric (Figure 9.23, c) loading, by the expressions:

$$\Delta_1 = Z_1 + Z_2, \qquad \Delta_2 = Z_1 - Z_2,$$

and the rotation angles of the nodes as:

$$\varphi_1 = -Z_3 + Z_4, \qquad \varphi_2 = Z_3 + Z_4.$$

These relations show that the unknown movements of nodes Δ_1, Δ_2, φ_1 and φ_2 ("old" unknowns) can be expressed in terms of the "new" unknowns Z_1, Z_2, Z_3, Z_4, which are grouped displacements of symmetrically located nodes. Introduction to the calculation of new unknowns leads to significant simplifications in the calculation. Unit bending moments diagrams from grouped displacements are divided only into symmetric or inverse-symmetric ones. Such diagrams have the property of mutual orthogonality, and therefore the system of canonical equations splits into two independent subsystems of equations, one of which contains only symmetric unknowns, and the second – inverse symmetric. The described method for calculating frames is called the method of grouping unknown displacements.

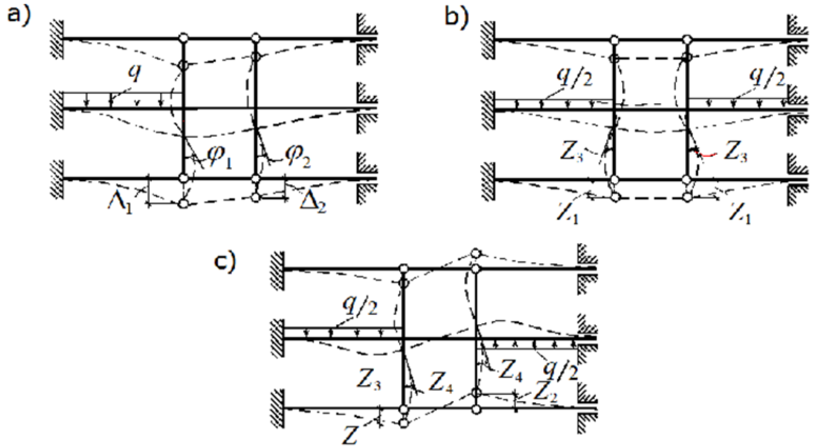


Figure 9.23

Note that if there is a bar on the frame symmetry axis (its position coincides with the axis of symmetry), then the symmetric unknowns do not cause bending moments in it. Therefore, the result of "multiplying" the symmetric diagram by inverse-symmetric one will be zero.

Coefficients and free terms in canonical equations are generalized reactions caused by the displacement of grouped (pair) unknowns. They are determined in a static or kinematic way.

E x a m p l e 1. It is necessary to build the final diagram of bending moments in a three-span symmetrical frame (Figure 9.24, a), assuming the rigidity in bending of all the bars is equal to ***EJ.***

The degree of kinematic indeterminacy of the frame is $n = n_a + n_l = 2 + 3 = 5$. The primary system and positive directions of grouped unknowns are shown in Figure 9.24, b. Unit diagrams of bending moments and the load diagram of bending moments are presented in Figures 9.24, c–g. Diagrams $\bar{M}_1$ and $\bar{M}_3$ are symmetric, and diagrams $\bar{M}_2$, $\bar{M}_4$ and $\bar{M}_5$ are inversely symmetric. Due to their orthogonality, the system of five canonical equations of the displacement method is divided into two subsystems. One contains symmetric unknowns:

$$\left.\begin{aligned} r_{11}Z_1 + r_{13}Z_3 + R_{1F} &= 0, \\ r_{31}Z_1 + r_{33}Z_3 + R_{3F} &= 0, \end{aligned}\right\}$$

and the second contains inversely symmetric unknowns:

$$\left.\begin{aligned} r_{22}Z_2 + r_{24}Z_4 + r_{25}Z_5 + R_{2F} &= 0, \\ r_{42}Z_2 + r_{44}Z_4 + r_{45}Z_5 + R_{4F} &= 0, \\ r_{52}Z_2 + r_{54}Z_4 + r_{55}Z_5 + R_{5F} &= 0. \end{aligned}\right\}$$

After determining unit and load reactions (we advise the reader to find them independently), these subsystems of equations will have the following form:

$$\left.\begin{aligned} 5.0Z_1 + \frac{1}{12}Z_3 + 90.0\frac{1}{EJ} &= 0, \\ \frac{1}{12}Z_1 + \frac{199}{288}Z_3 + 6.0\frac{1}{EJ} &= 0, \end{aligned}\right\}$$

$$\left.\begin{aligned} 5.0Z_2 - \frac{1}{12}Z_4 - \frac{2}{3}Z_5 &= 0, \\ -\frac{1}{12}Z_2 + \frac{199}{288}Z_4 - \frac{2}{9}Z_5 - 6.0\frac{1}{EJ} &= 0, \\ -\frac{2}{3}Z_2 - \frac{2}{9}Z_4 + \frac{2}{9}Z_5 - 14.0\frac{1}{EJ} &= 0. \end{aligned}\right\}$$

Their solutions are:

$$Z_1 = -17.8912\frac{1}{EJ}\ \text{rad}, \qquad Z_2 = 41.1114\frac{1}{EJ}\ \text{rad},$$

$$Z_3 = -6.5257\frac{1}{EJ}\ \text{m}, \qquad Z_4 = 108.445\frac{1}{EJ}\ \text{m}, \qquad Z_5 = 294.78\frac{1}{EJ}\ \text{m}.$$

The final diagram of bending moments is based on the expression:

$$M = M_F + \bar{M}_1 Z_1 + \bar{M}_2 Z_2 + \bar{M}_3 Z_3 + \bar{M}_4 Z_4 + \bar{M}_5 Z_5\,.$$

It is depicted in Figure 9.24, i.

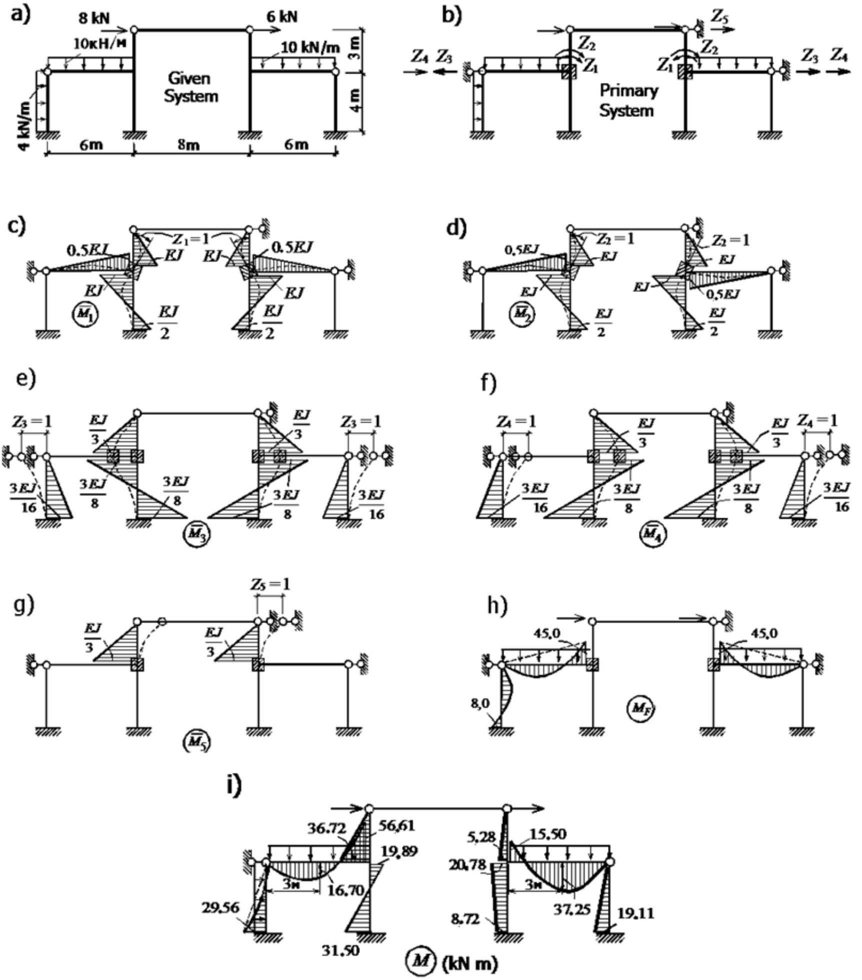


Figure 9.24

Analyzing the subsystems of equations recorded in this example, we can draw the following conclusions.

1. If a symmetric exposure has an effects on a symmetrical frame, then the free terms in the system of equations with inversely symmetric

unknowns will be equal to zero. Therefore, inversely symmetric unknowns will also be equal to zero (from the solution of the system of homogeneous equations).

2. If the external action is inversely symmetric, then the symmetric unknowns become zero.

As a consequence of the method of grouping unknowns, it should be noted that the calculation of a symmetric system for a symmetric or inversely symmetric load can be performed for one half of the design scheme. Depending on the loading in the second half of the system, the distribution of forces will be symmetrical or inversely symmetric compared to the first half.

In particular, if the axis of symmetry crosses a certain bar, then in this section it is necessary to set movable pinching, when a load is symmetrical. For example, for the frame (Figure 9.22, a), the corresponding "half frame" is shown in Figure 9.25, a. Under the action of the inverse symmetric load in such section, the deformed axis of the bar has an inflection point and, in addition, the vertical displacement of the section (in the direction of the axis of symmetry) is zero. Therefore, in the design scheme of the "half frame" in the indicated section, a hinged movable support is placed (Figure 9.25, b).

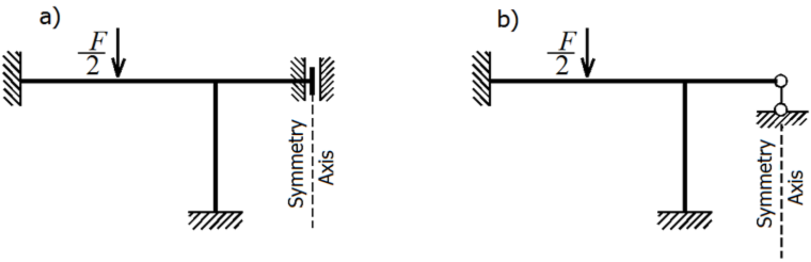


Figure 9.25

9.9. Calculating Frames Subjected to Thermal Effects

The calculation is carried out to change the temperature of the system with respect to the temperature of its initial state. Accepting the linear law of temperature change along the height of the cross section of the bar, the thermal effect can be represented as the sum of the symmetric and inverse symmetric components of this effect.

Suppose, for example, that a bar has a symmetric cross section of height d (Figure 9.26, a) and $t_1 > t_2$, i. e., the lower fibers of the bar are "warm", but the upper ones are "cold".

Let us decompose this effect into a symmetric and inversely symmetric. At symmetric exposure the bar is uniformly heated (Figure 9.26, b). The temperature of the upper and lower fibers is the same and equal to

$$t = \frac{t_1 + t_2}{2}.$$

The elongation of the bar in this case is equal to $\alpha t l$. At inversely symmetric exposure (Figure 9.26, c), the temperature of the upper fiber is equal to $-\frac{t_1 - t_2}{2} = -\frac{t'}{2}$, and the bottom to $\frac{t'}{2}$. With inversely symmetric heating, the temperature along the axis of the bar is zero. The bar from such exposure does not elongate, but only bends. The value of the displacement of any of its points is determined according to the rules set out in section 7.8.

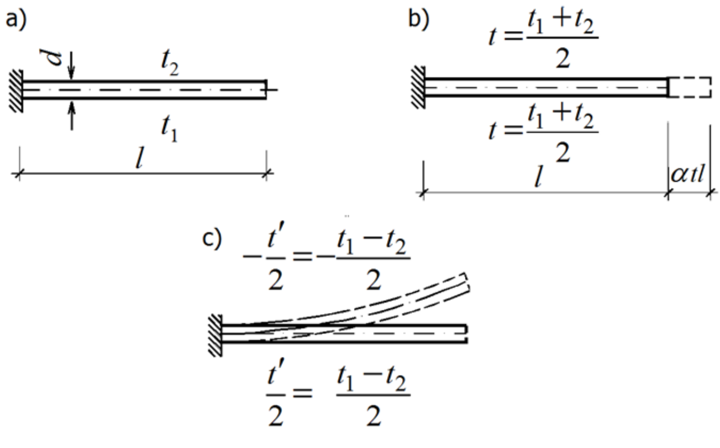


Figure 9.26

A similar decomposition of the thermal effect can be done for the bars with the conditions for the fastening of their ends corresponding to the fastening of the bars in the primary system of the displacement method.

With a symmetric distribution of temperatures, due to the elongation (shortening) of the bars, the nodes of the primary system move, which leads to mutual displacements of the ends of the bars in the transverse direction and causes their bending deformation.

In the case of inverse symmetric temperature distribution, the nodes of the primary system are not displaced, but since the bonds at the ends of the bars impede their free movement, forces appear in each of them. Diagrams of moments for such bars are presented in table 9.1 (lines 5, 10, 15). The technique of their construction is given in section 9.2 (example 5).

The canonical equations for calculating the frames for temperature change are as follows:

$$\left.\begin{aligned} r_{11}Z_1 + r_{12}Z_2 + \cdots + r_{1n}Z_n + R_{1t} = 0, \\ r_{21}Z_1 + r_{22}Z_2 + \cdots + r_{2n}Z_n + R_{2t} = 0, \\ . \qquad . \qquad . \qquad . \qquad . \\ r_{n1}Z_1 + r_{n2}Z_2 + \cdots + r_{nn}Z_n + R_{nt} = 0. \end{aligned}\right\} \tag{9.11}$$

To determine free terms $R_{1t}, R_{2t}, \cdots, R_{nt}$ of the canonical equations, as follows from the previous reasoning, it is necessary to construct diagrams of bending moments in the primary system: (M_t') due to symmetric termal action and (M_t'') due to inversely symmetric action. Using them, we find that:

$$R_{it} = R_{it}' + R_{it}'', \qquad i = \overline{1, n},$$

where R_{it}', R_{it}'' are the reactions in the i-th additional bond (link) caused by these influences.

The final diagrams of bending moments is built according to the formula

$$M = M_t' + M_t'' + \overline{M}_1 Z_1 + \overline{M}_2 Z_2 + \cdots + \overline{M}_n Z_n .$$

E x a m p l e 1. Let us build the final diagram of the bending moments in the frame (Figure 9.27, a) from the specified heat exposure, taking the stiffness of the bars the same and equal to 60 MN·m^2, the height of the cross section d = 0.6·m, the coefficient of thermal linear expansion α = $1.2\cdot10^{-5}\cdot(°C)^{-1}$.

The kinematic indeterminacy of the frame is two. Figure 9.27, b shows the symmetric temperature distribution for each bar, and Figure 9.27, c – inversely symmetric.

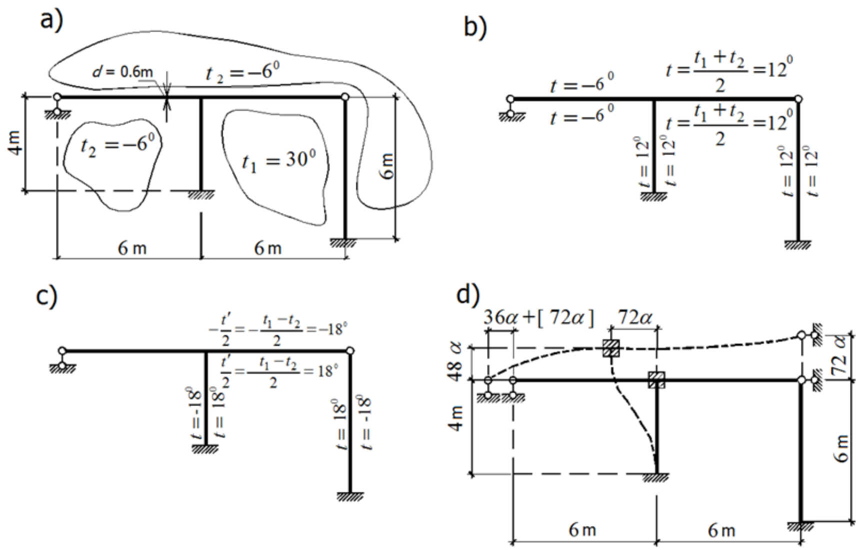


Figure 9.27

The system of canonical equations in this case can be written as:

$$\left.\begin{aligned} r_{11}Z_1 + r_{12}Z_2 + R_{1t} = 0, \\ r_{21}Z_1 + r_{22}Z_2 + R_{2t} = 0. \end{aligned}\right\}$$

Due to the symmetric effect of temperatures, the plotting of moment diagram M_t' must begin by determining the elongation of each bar according to the formula $\Delta = \alpha t l$. Then it is necessary to depict the new position of the nodes and the deformed position of the bars in the primary system (Figure 9.27, d). Knowing the mutual displacements of the ends of each bar in the direction perpendicular to its axis and using the data from Table. 9.1 (lines 1, 6, 11), it is possible to make a diagram of bending moments. Final diagram M_t' is shown in Figure 9.28, a.

In this example, to build diagram M_t'' (Figure 9.28, b), lines 5 and 10 from table 9.1 are used. On each bar of the frame, the stretched fibers are more "cold". It is from this side of the bar that the bending moments diagram is located.

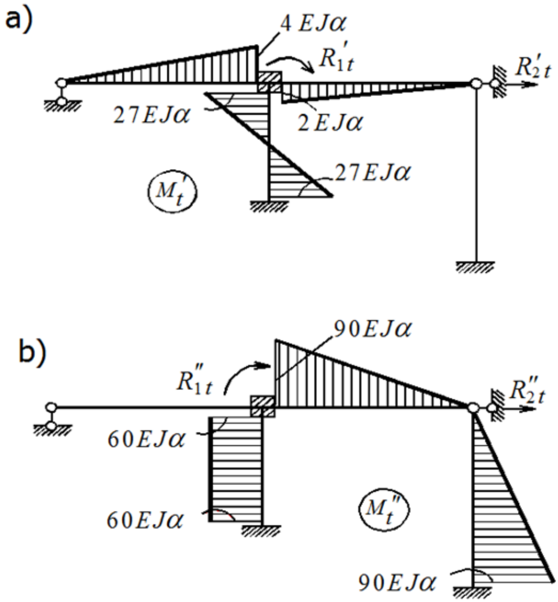


Figure 9.28

Unit diagrams of moments are shown in Figure 9.29, a, b.

The reactive forces in additional links having been determined, the system of canonical equations can be written in numerical form:

$$\left.\begin{array}{rcrcrl} 2Z_1 & - & 0.375Z_2 & + & 3\alpha & =0, \\ -0.375Z_1 & + & 0.2013888Z_2 & - & 28.5\alpha & =0. \end{array}\right\}$$

Its solution is: $Z_1 = 38.4636\alpha$ rad; $Z_2 = 213.1391\alpha$ m.

The final bending moments diagram is shown in Figure 9.30. In parentheses there are the ordinates of the moments (in kN·m) at EJ = $= 60$ MN·m^2 and $\alpha = 1.2\ 10^{-5}$ deg^{-1}.

The correctness of the plot of bending moments is checked using the equilibrium conditions of any fragments of the frame, in particular, the nodes of the frame. As a rule, this check is enough to conclude that plot M is correct.

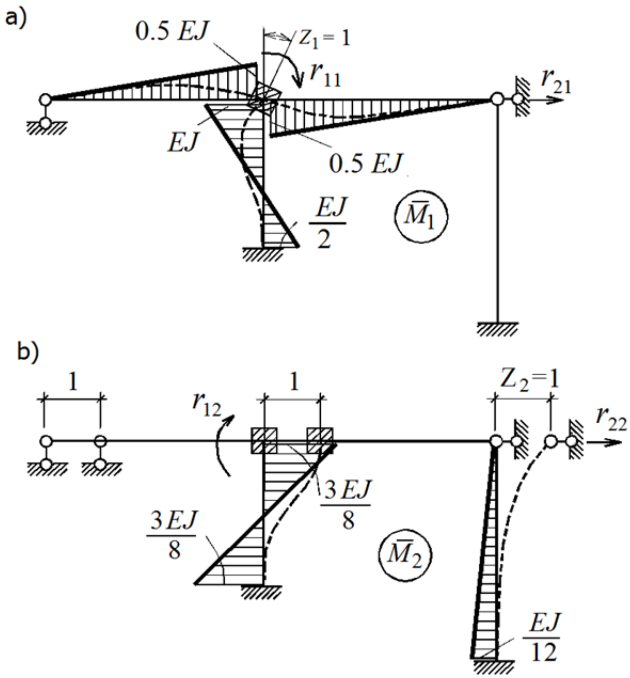


Figure 9.29

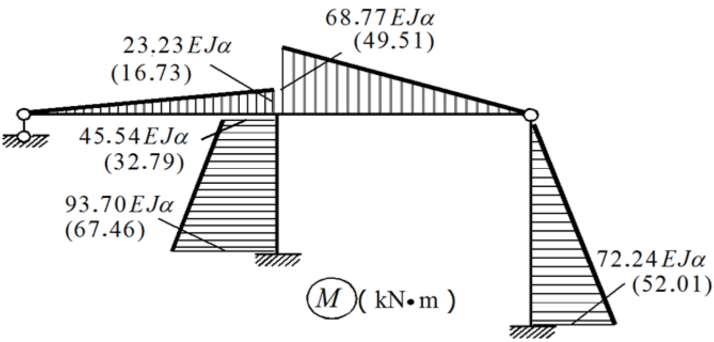


Figure 9.30

However, in addition, another check can be used: the result of “multiplying” the total unit diagram of the moments of the method of displacement by the final diagram should be zero, that is:

$$\sum\int\frac{\bar{M}_S M dx}{EJ}=0\,.$$

9.10. Calculating Frames Subjected to Settlement of Supports

A distinctive feature of the calculation of frames at the given support settlements is associated with constructing the diagram of the bending moments in the primary system of the displacement method due to such displacements. In the future we will denote this diagram by M_c. To do this, it must be remembered that additional floating links in the primary system prevent only the rotation of rigid nodes, while they allow linear displacements of nodes. Therefore, the effect of linear (horizontal or vertical) displacement of any support can be extended on too many frame elements adjacent to the displaced post or the displaced crossbar.

When constructing diagram M_c, it is recommended to use the principle of independence of the action of forces. First, it is necessary to construct the bending moments diagrams caused by the given displacements of each support individually. Then the total summarized diagram of bending moments M_c must be constructed, with the help of which free terms R_{ic} are determined. As a result, a system of canonical equations is formed:

$$R\vec{z}+\vec{R}_c=0\,.$$

The following calculation algorithm remains the same as when calculating frames for force impact.

The main verification of the final bending moments diagram is reduced to checking that the equilibrium conditions of the nodes and other parts of the frame are satisfied.

As in the calculation of the thermal effect, the result of the “multiplication” of the total unit moments diagram $\bar{M}_S$ of the displacement method on the final moments diagram should be zero, that is:

$$\sum\int\frac{\bar{M}_S M\,dx}{EJ}=0.$$

Note that no matter what method was used to construct the final diagram of bending moments, to check its correctness you can apply the kinematic check used in the force method. In particular, when calculating frames for support displacement, the result of "multiplying" the total unit moment diagram of the force method by the final moment diagram should be equal to the sum of the free terms of the canonical equations of the force method, taken with the opposite sign:

$$\sum\int\frac{\bar{M}_S M\,dx}{EJ}=-\sum\Delta_{ic},$$

where $\bar{M}_S$ is the total unit diagram in the primary system of the method of forces.

E x a m p l e 1. Let us consider the features of calculating the frame, the support of which is displaced as shown in Figure 9.31, a. Let c_1 = = 0.02 m, c_2 = 0.04 m and c_3 = 0.1 rad. The bending stiffnesses of all bars are assumed the same.

The primary system and the primary unknowns of the displacement method are shown in Figure 9.31, b.

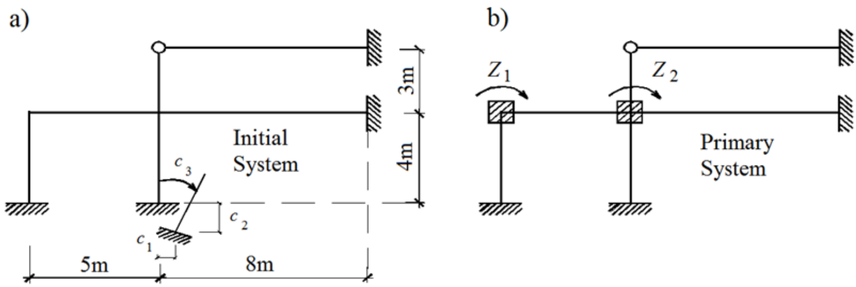


Figure 9.31

Unit bending moments diagrams are shown in Figures 9.32, a, b.

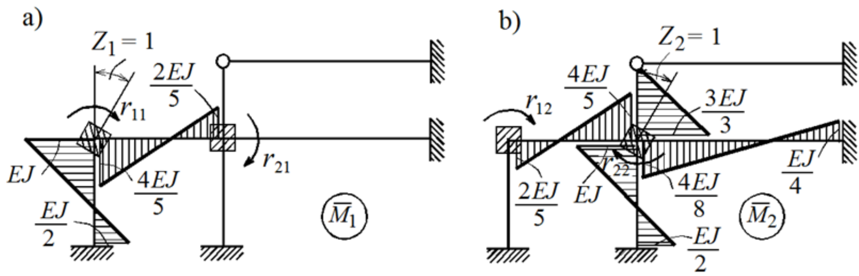


Figure 9.32

Figures 9.33, a, b, c show the diagrams of the bending moments caused by the displacements of the support connections, respectively, by c_1, c_2, and c_3.

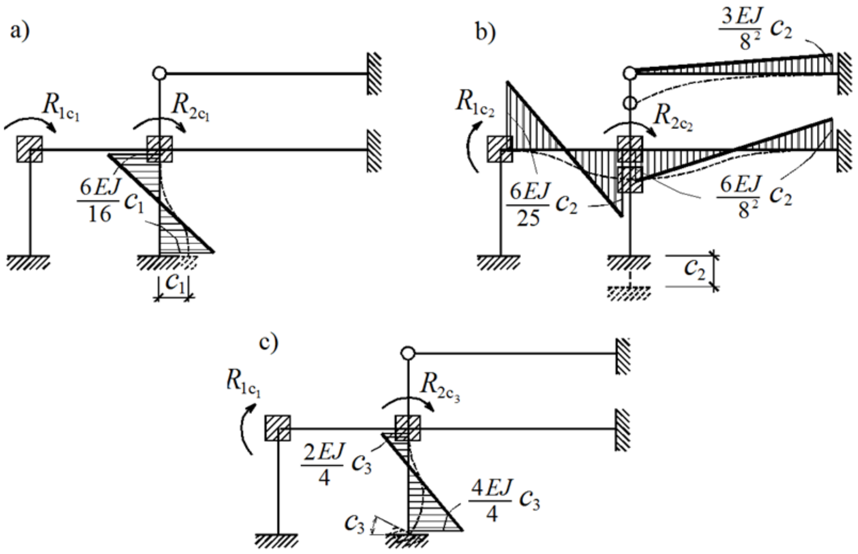


Figure 9.33

The construction of these diagrams, as well as unit ones, is performed using the data in Table 9.1. The total diagram M_c (not shown in this example) is constructed by the expression:

$$M_c = M_{c_1} + M_{c_2} + M_{c_3}.$$

The canonical equations have the form:

$$\left.\begin{aligned} r_{11}Z_1 + r_{12}Z_2 + R_{1c} = 0, \\ r_{21}Z_1 + r_{22}Z_2 + R_{2c} = 0, \end{aligned}\right\}$$

where

$$R_{1c} = R_{1c_1} + R_{1c_2} + R_{1c_3}, \qquad R_{2c} = R_{2c_1} + R_{2c_2} + R_{2c_3}.$$

After determining the values of the coefficients and free terms, we solve the system of equations and find:

$$Z_1 = 0.00905 \text{ rad}; \qquad Z_2 = -0.01675 \text{ rad}.$$

The final diagram of the bending moments is shown in Figure 9.34.

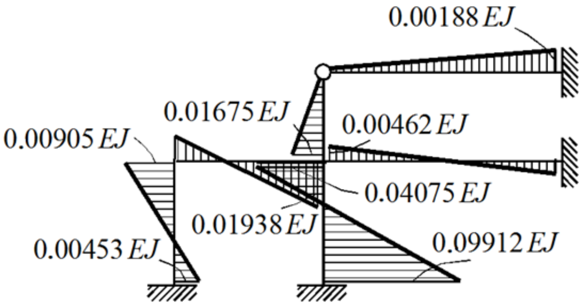


Figure 9.34

9.11. Constructing Influence Lines for Efforts

The methodology for constructing the influence lines of efforts in statically indeterminate systems is reduced to the implementation of the formulas by which they are calculated. When calculating frames by the displacement method, the force in cross section *"k"* is calculated by the formula:

$$S_k = S_{kF} + \sum_{i=1}^{n} \bar{S}_{ki} Z_i.$$

The quantities S_{kF} and Z_i given in this expression to the right of the equal sign are variables depending on the position of the external force $F=1$, but quantity $\bar{S}_{ki}$ is the constant force in cross section "k" of the primary system caused by the offset of the *i*-th additional link by one. Therefore, in relation to the problem of constructing influence lines, this record can be represented as:

$$\text{Inf. Line } S_k = \text{Inf. Line } S_{kF} + \sum_{i=1}^{n} \bar{S}_{ki} \text{ (Inf. Line } Z_i\text{)}.$$

In the last expression the designation "Inf. Line S_{kF}" is the influence line for effort S_k in the primary system of the displacement method. The name of effort S_{kF} in this case can be replaced by S_k^0.

The construction of this influence line does not cause any particular difficulties. Indeed, in the primary system, the ends of each bar in the nodes are pinched or pivotally supported and therefore the load located on it does not affect the forces in adjacent bars. As a result, the influence line of S_{kF} will have ordinates that are not equal to zero, only on the bar to which section "k" belongs. Determining S_k according to the table 9.1 for various positions of force $F=1$, we will construct Inf. Line S_k^0 (Inf. Line S_{kF}).

It is more complex to construct influence lines Z_i. Building them based on a static method requires determination of the values of the primary unknowns for various positions of the unit force $F=1$.

Consider the following example. The frame shown in Figure 9.35, a, is once kinematically indeterminable. The canonical equation

$$r_{11}Z_1 + R_{1F} = 0$$

for $F=1$ can be rewritten in the form

$$r_{11}Z_1 + r_{1F} = 0.$$

Consequently,

$$z_1 = -\frac{r_{1F}}{r_{11}}.$$

Diagram M_1 in the primary system is shown in Figure 9.35, b. From the equilibrium equation

$$\sum M_B = 0$$

it follows that

$$r_{11} = \frac{11EJ}{l}.$$

In the loading state of the primary system at positions of force $F = 1$ to the left of node B (Figure 9.35, c) the free term is equal to:

$$r_{1F} = \frac{l}{2}v\left(1 - v^2\right).$$

The corresponding angle of rotation Z_1 will be determined by the expression:

$$Z_1 = -\frac{l^2}{22EJ}v\left(1 - v^2\right).$$

By setting variables v and $u = 1 - v$ values from 0 to 1, we calculate Z_1 and construct a line of influence in length AB.

With the location of force $F = 1$ on the console section of the frame (Figure 9.35, d) we get

$$r_{1F} = -1 \cdot x \text{ and } Z_1 = \frac{x}{11EJ}l.$$

Inf. Line Z_1 is shown in Figure 9.35, d.

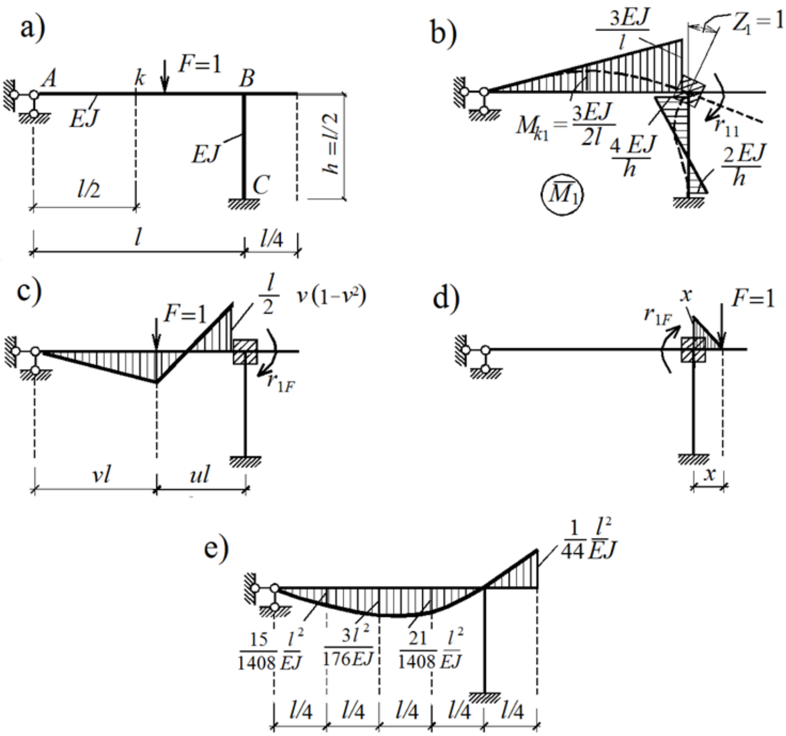


Figure 9.35

The shape of the constructed influence line can be checked using the kinematic method. Section 7.11 shows that to construct an influence line for the displacement of *i*-th cross section, it is necessary to apply a unit force in this section and plot the displacements diagram due to this force. The position of the bended axes of the bars along which force $F = 1$ moves will correspond to the outline of the influence line for the investigated displacement. This method is based on the theorem on reciprocity of displacements.

In this example, to build an influence line of the angle of rotation of node B (Inf. Line Z_1), it is necessary to load this node by a unit moment in the positive direction of Z_1 and show the position of the curved axes of the bars (Figure 9.36).

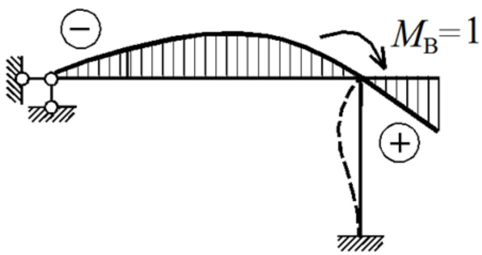


Figure 9.36

The ordinates of the displacement diagram thus obtained, counted from the initial position of the bars in the direction of the force $\boldsymbol{F} = \mathbf{1}$, are considered positive.

Their numerical values, if necessary, can be found according to the well-known rules of structural mechanics.

To build, for example, Inf. Line M_k in the given system (Figure 9.35, a) it is necessary to pre-build Inf. Line M_k^0 in the primary system. We restrict ourselves to considering the position of the unit force on the beam AB in three characteristic sections (Figure 9.37, a).

Having determined for each position of the force the value of the bending moment in cross section "k", we construct Inf. Line M_k^0 (Figure 9.37, b).

Next, using the expression:

$$\text{Inf. Line } M_k = \text{Inf. Line } M_k^0 + \bar{M}_{k1} \text{ (Inf. Line } Z_1\text{)},$$

and determining that (Figure 9.35, b)

$$\bar{M}_{k1} = -\frac{3EJ}{2l},$$

we obtain Inf. Line M_k (Figure 9.38).

It is clear that for frames with a large number of unknowns, the amount of computation for constructing influence lines increases significantly. Therefore, a practical solution to the problem of constructing the influence lines for efforts in a frame is reduced to calculating it with the help of certified software systems for many unit loads and compiling an appropriate influence matrix.

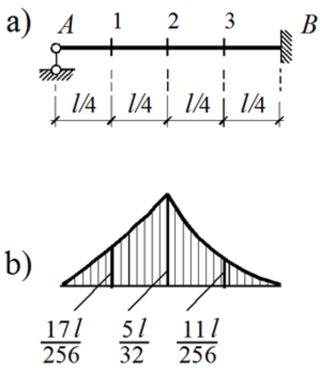


Figure 9.37

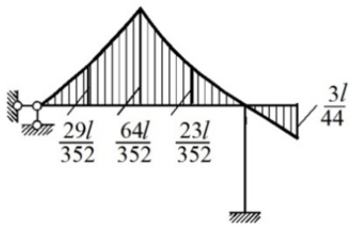


Figure 9.38

It is known that it has the form:

$$L_S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ . & . & . & . \\ S_{i1} & S_{i2} & \cdots & S_{in} \\ . & . & . & . \\ S_{m1} & S_{m2} & \cdots & S_{mn} \end{bmatrix}.$$

By definition, S_{ik} is the force in the i-th section of the structure caused by a unit force applied in section k. Elements of the i-th row of the matrix L_S give the ordinates of the influence line for effort S_i. In order to find the elements of the k-th column of the influence matrix, it is necessary to calculate the given structure for loading it by force $F_k = 1$. The number of such unit loads is n.

The following example will explain the features of calculating and plotting the influence line for efforts using the matrix of the influence of efforts.

The design diagram of the frame is shown in Figure 9.39, a. The shape of any influence line in the area between adjacent nodes can be represented by the values of the ordinates of efforts in three equally spaced sections. Therefore, we will calculate the frame by loading it with forces $F = 1$ applied sequentially in each characteristic section between the nodes and compose, for example, the matrix of the influence of bending moments L_M. This matrix has the form:

$L_M =$

1.6647	0.9135	0.3305	–0.1052	–0.1042	–0.0511	0.0210	0.0240	0.0150
0.8293	1.8269	0.6611	–0.2103	–0.2083	–0.1022	0.0421	0.0481	0.0300
–0.0060	0.2104	0.9916	–0.3155	–0.3125	–0.1532	0.0631	0.0721	0.0451
–0.2654	–0.4247	–0.3716	1.0196	0.2778	0.0220	0.0070	0.0080	0.0050
–0.1302	–0.2083	–0.1823	0.5208	1.5278	0.5208	–0.1823	–0.2083	–0.1302
0.0050	0.0080	0.0070	0.0220	0.2778	1.0196	–0.3716	–0.4247	–0.2654
0.0451	0.0721	0.0631	–0.1532	–0.3125	–0.3155	0.9916	0.2404	–0.0060
0.0300	0.0481	0.0421	–0.1022	–0.2083	–0.2103	0.6611	1.8269	0.8293
0.0150	0.0240	0.0210	–0.0511	–0.1042	–0.1052	0.3305	0.9135	1.6647

According to the second and fifth rows of matrix L_M, Inf. Line M_2 (Figure 9.39, b) and Inf. Line M_5 (Figure 9.39, c) are constructed. Since the stiffness of the rods in tension-compression in this calculation is assumed to be infinity, i.e. their longitudinal deformations are neglected, then at the points corresponding to nodes A and B, the ordinates of the influence lines are zero.

Of course, it is advisable to perform calculations of such and more complex systems for many loads with the help of software systems available in design organizations.

In the same way it would be possible to compose matrices of the influence of transverse or longitudinal forces and construct the necessary influence line for these efforts. Figure 9.39, d shows the influence line for Q_5.

The calculation results also allow one to obtain influence matrices of displacements. Let us pay attention to the line of influence of vertical displacement of section 2 (Figure 9.39, f) and influence line for the angle of rotation of node B (Figure 9.39, e). Their shape is consistent with the recommendations of section 7.11: to obtain the first one should use force $\boldsymbol{F} = 1$ applicated in section 2; in the second case, at point B, moment $\boldsymbol{M} = 1$ is applied (directed counterclockwise). The numerical values of the ordinates correspond to $\boldsymbol{EJ} = 13.5$ МН · м^2.

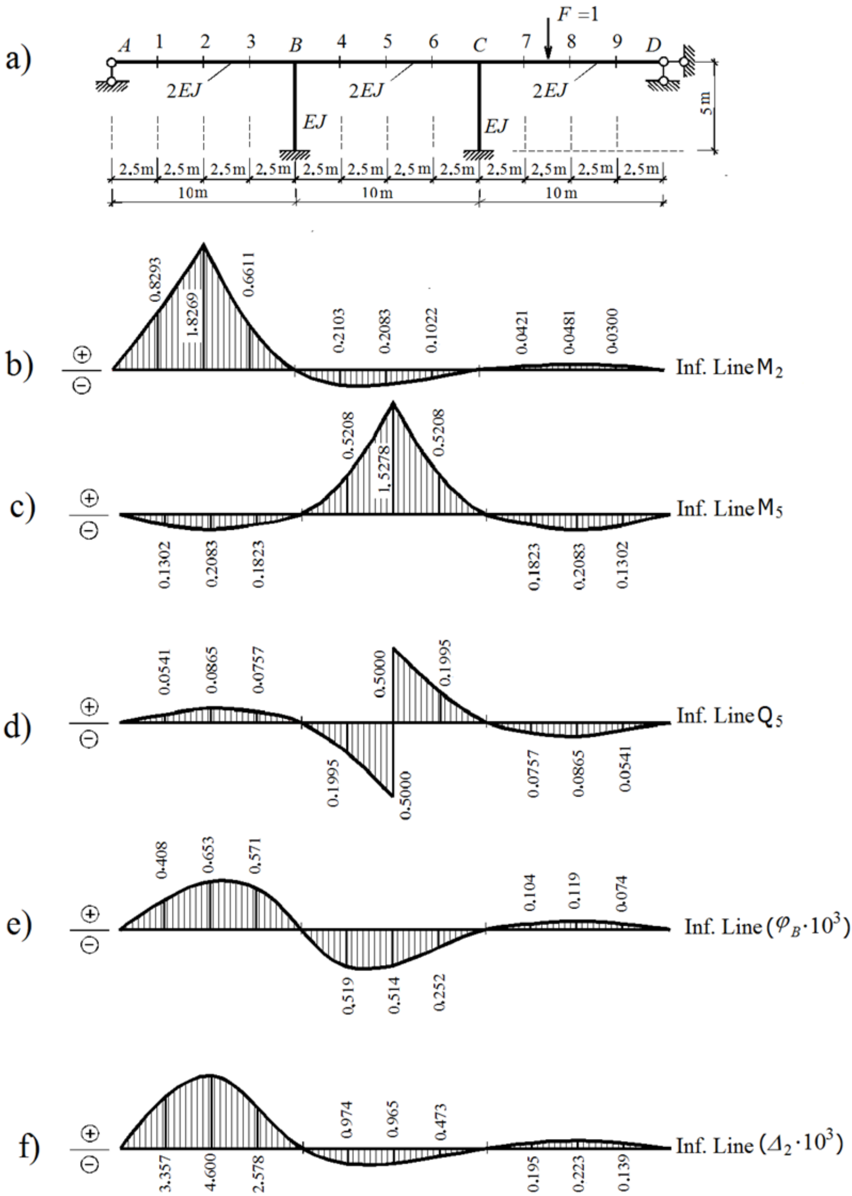

a)
F =1
A 1 2 3 B 4 5 6 C 7 8 9 D
2EJ 2EJ 2EJ
EJ EJ
5m
2.5m 2.5m 2.5m 2.5m 2.5m 2.5m 2.5m 2.5m 2.5m 2.5m 2.5m 2.5m
10m 10m 10m
b)
0.8293 1.8269 0.6611
0.2103 0.2083 0.1022
0.0421 0.0481 0.0300
Inf. Line M2
c)
0.5208 1.5278 0.5208
0.1302 0.2083 0.1823
0.1823 0.2083 0.1302
Inf. Line M5
d)
0.0541 0.0865 0.0757
0.5000 0.1995
0.1995 0.5000
0.0757 0.0865 0.0541
Inf. Line Q5
e)
0.408 0.653 0.571
0.104 0.119 0.074
0.519 0.514 0.252
Inf. Line (φB·10³)
f)
0.974 0.965 0.473
3.357 4.600 2.578
0.195 0.223 0.139
Inf. Line (Δ2·10³)

Figure 9.39

9.12. Calculating Frames Taking into Account Longitudinal Deformation of the Bars

As noted in Section 9.1, each rigid node of a plane frame has three degrees of freedom, each hinged node – two. For this reason, the degree of kinematic indeterminacy of the frames, in the calculation of which the longitudinal deformations of the bars are taken into account, is much higher than the frames, in the calculation of which the longitudinal deformations are neglected.

In order to form the primary system of the displacement method, three additional links are superimposed on each rigid node of a given system, and two linear links on each hinged one. To determine the coefficients and free terms of the canonical equations, it is necessary to construct diagrams of bending moments and longitudinal forces in the primary system, caused by unit displacements of additional links and given exposures.

The values of the coefficients and free terms are found by the static method from the equilibrium equations of the nodes of the main system.

The same values of the coefficients of the canonical equations of the displacement method can be found with kinematic method by the formula:

$$r_{ik} = \sum \int \frac{\overline{M}_i \overline{M}_k dx}{EJ} + \sum \int \frac{\overline{N}_i \overline{N}_k dx}{EA}.$$

The second term in this expression is calculated in the same ways as the first, in particular, for example, by "multiplying" the corresponding diagrams of longitudinal forces.

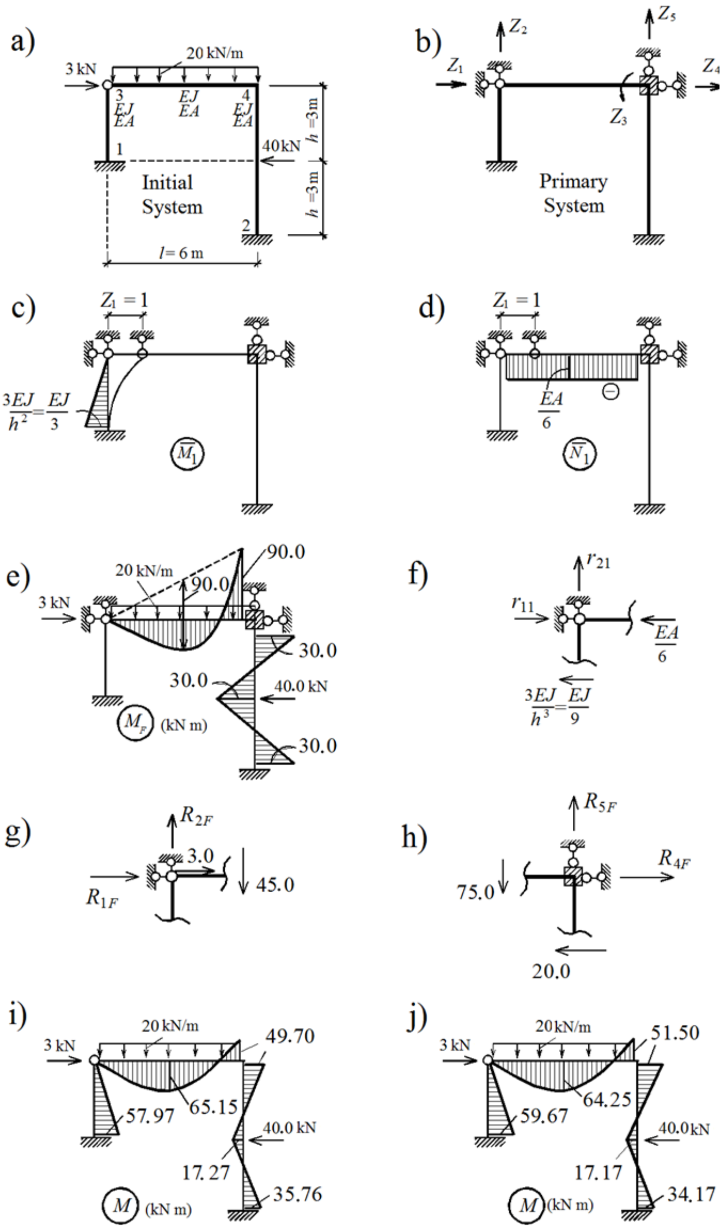
The influence of this term on the value of r_{ik}, as follows from the above expression, increases with a decrease in the stiffness of the bars on tension-compression.

Some features of calculating the coefficients and distribution of bending moments will be shown on the example of the frame shown in Figure 9.40 a.

The primary system and the positive directions of the primary unknowns are shown in Figure 9.40, b. We restrict ourselves to considering diagrams $\overline{M}_1, \overline{N}_1$ and $\overline{M}_F$ (Figures 9.40, c–e).

From the equilibrium condition of node 3 (Figure 9.40, f)

$$\left(\sum X = 0\right) \quad \text{we find} \quad r_{11} = \frac{EJ}{9} + \frac{EA}{6}.$$



a)
20 kN/m
3 kN
3
EJ
EA
EJ
EA
4
EJ
EA
40 kN
h =3m
h =3m
1
Initial
System
2
l= 6 m
b)
Z2
Z5
Z1
Z4
Z3
Primary
System
c)
Z1 = 1
3EJ/h² = EJ/3
M1
d)
Z1 = 1
EA/6
N1
e)
90.0
20 kN/m
90.0
3 kN
30.0
30.0
40.0 kN
30.0
MF (kN m)
f)
r21
r11
EA/6
3EJ/h³ = EJ/9
g)
R2F
3.0
45.0
R1F
h)
R5F
R4F
75.0
20.0
i)
20 kN/m
49.70
3 kN
57.97
65.15
40.0 kN
17.27
35.76
M (kN m)
j)
20 kN/m
51.50
3 kN
59.67
64.25
40.0 kN
17.17
34.17
M (kN m)

Figure 9.40

Naturally, the same value r_{11} will be obtained by the kinematic method:

$$r_{11} = \frac{1}{EJ}\frac{1}{2}\frac{EJ}{3}\cdot 3\cdot\frac{2}{3}\frac{EJ}{3} + \frac{1}{EA}\frac{EA}{6}\cdot 6\cdot\frac{EA}{6} = \frac{EJ}{9} + \frac{EA}{6}.$$

To determine the free terms of the canonical equations, one should use the distribution of forces at the nodes shown in Figures 9.40, g, h, or use the kinematic method. In the latter case:

$$R_{iF} = -\sum\int\frac{\bar{M}_i M_F^0 dx}{EJ} - \sum\int\frac{\bar{N}_i N_F^0 dx}{EA},$$

where $\bar{M}_i, \bar{N}_i$ – functions of bending moments and longitudinal forces from unit displacements of nodes in the primary system of the displacement method;

M_F^0, N_F^0 – functions of bending moments and longitudinal forces from the given load in the primary system of the force method.

The influence of longitudinal deformations on the distribution of internal forces in the frame can be judged by the final bending moments diagrams constructed at the ratio

$$\frac{EA\cdot h^2}{EJ} = 10,$$

where $\boldsymbol{h}$ = 1 m (Figure 9.40, i), and at $EA \to \infty$ (there are no longitudinal deformations) (Figure 9.40, j). At

$$\frac{EA\cdot h^2}{EJ} < 10,$$

the calculation results will be even more different from those that correspond to the calculation option with $EA \to \infty$.

An increase in the flexibility of the bars under tension-compression leads to an increase in the displacements of the nodes, and therefore, the calculation of such frames according to an undeformed scheme should be considered as approximate.

THEME 10. SIMULTANEOUS APPLICATION OF THE FORCE AND DISPLACEMENT METHODS. MIXED METHOD

10.1. Force and Displacement Methods in Comparison

Both the force method and the displacement method have its advantages and disadvantages. Each of them, taking into account the assumptions used in the calculation, is accurate. In both methods, it is also possible to take into account the influence of longitudinal and shear deformations in addition to bending deformations. Which one should be used for calculation?

With manual calculations, the search for the best method for calculating of the given system is reduced, in most cases, to the search for a calculation variant with the least laboriousness. Most often the choice of one or another method depends on the number of unknowns.

As a rule the best way to do the calculation of frames, the nodes of which do not have linear mobility is by the displacement method. Diagrams of effort are easy to construct, have local character and, because of this, the system of canonical equations is rarefied. However, when longitudinal deformations are taken into account the number of displacement method unknowns increases significantly.

The choice of a rational primary system of the method of forces and the construction of diagrams are associated with a more complex logic of understanding the structure of the system. The operation to calculate the coefficients and free terms of the canonical equations is also quite time-consuming. In the displacement method, this part of the calculations, carried out, for example, in a static way, is less time-consuming. As an advantage of the force method, we note that the degree of static indeterminacy of a given system does not depend on whether or not the influence of longitudinal deformations is taken into account in the calculation.

The above remarks on the methods discussed are a qualitative characteristic of them. Note, in addition, that in each particular case, the engineer has the right to choose any of them, guided by their own level of knowledge of these calculation methods.

The decision to choose a method for automated computing is associated not so much with the computational procedures for each of them, but with the features of the primary system. The logic of automating the process of choosing the primary system of the displacement method is simpler. The ideas of the displacement method are widely used in the development of software systems for calculating and designing building structures.

10.2. Force and Displacement Methods. Simultaneous Application

To choose a rational calculation method an engineer must deep understand the main principles of the force and displacement methods. In particular, joint application of these methods is possible for the calculation of both symmetric and asymmetric systems.

Let us first consider the features of the calculation of **symmetric systems**. The load applied to such a system can always be decomposed into symmetrical and inverse-symmetric (otherwise, skew-symmetric) components. As a rule, it turns out that it is convenient to calculate a frame under the symmetric load component by one method, for example, by the displacement method, and it is convenient to calculate the same frame under the inverse-symmetric load component by the other method, for example, by the force method. The final result of calculating the frame for a given load is obtained by summing the results of its calculation on both load components. Now let us consider an example.

The frame (Figure 10.1, a) has four unknowns by the displacement method (the primary system and the primary unknowns are shown in Figure 10.1, b) and four unknowns by the force method (Figure 10.1, c). The symmetric and inverse-symmetric components of the given load are presented in Figures 10.1, d, e. The calculation of the frame due to the action of the symmetrical load is performed by the displacement method, since in this case the only unknown, not equal to zero, will be unknown Z_1. To calculate the frame on the inverse-symmetric load, we use the force method, because in this case only X_3 will be non-zero.

The bending moment diagrams corresponding to the action on the frame of the symmetrical and inverse-symmetric loads are shown in Figures 10.1, f, g. And the final diagram of bending moments is shown in Figure 10.1, h.

The presented version of the frames calculation in the educational literature is sometimes called the combined method of calculation.

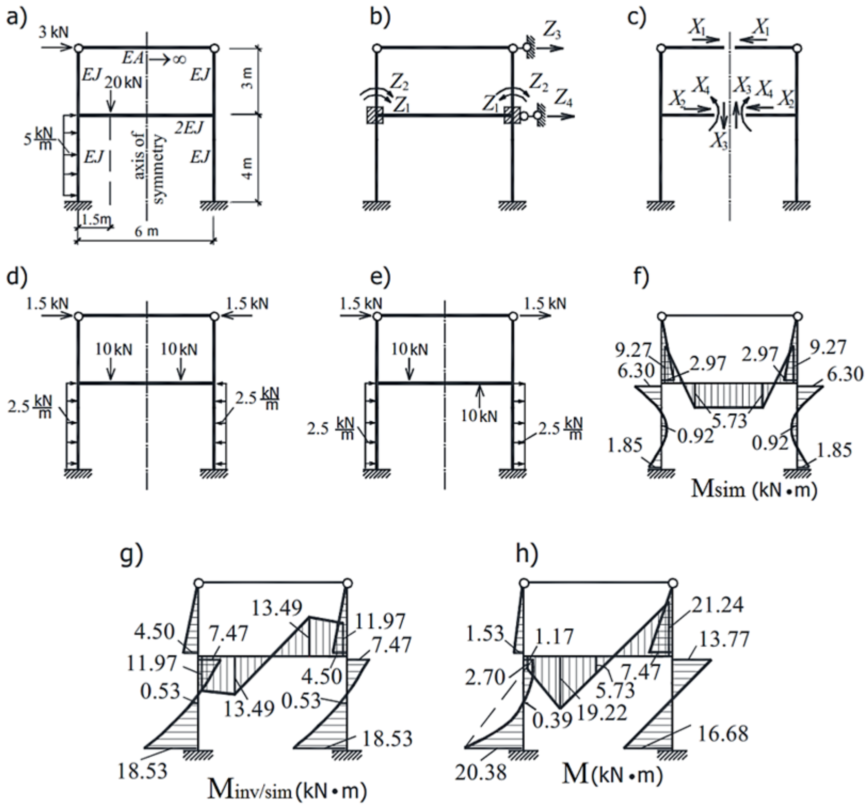


Figure 10.1

The combined application of the force and the displacement methods is also possible in the calculation of **non-symmetric systems.**

In order to reduce the number of unknowns, the primary systems of the force method or the displacement method can be adopted, respectively, statically or kinematically indeterminate. The same method is used to calculate both the given system and the primary system.

Moreover, if the calculation is performed by the force method, then the primary statically indeterminated system is chosen so that it may be easly calculated by the displacement method. In this case, the force method is the main calculation method, and the displacement method is the auxiliary method.

If the displacement method is taken as the main method, then the kinematically indeterminate fragment of the primary system is calculated by the force method (auxiliary method).

The features of such calculation will be explained in the following examples.

For the frame (Figure 10.2, a), we will take the displacement method as the main calculation method, and select the primary system according to the variant shown in Figure 10.2, b, i.e. we will calculate the given frame as a twice kinematically indeterminate system.

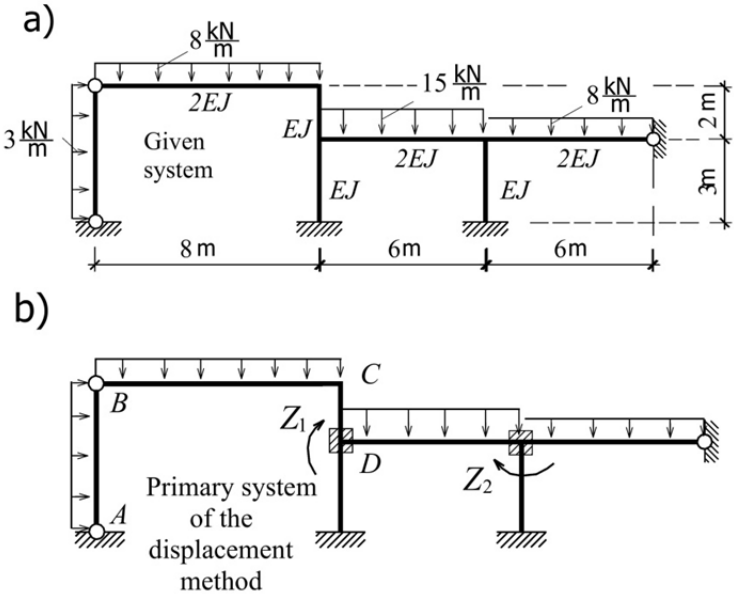


Figure 10.2

To construct diagrams of bending moments caused by the given loads and by the unit values of unknowns, first it is necessary to first calculate a statically indeterminate fragment $ABCD$ on the action of the given load and the rotation of the support constraint at point D through angle $Z_1 = 1$. The frame $ABCD$ is statically indeterminate once (by the displacement method the number of unknowns is equal to two). Therefore, its calculation is performed by the force method, which in this version of its use is considered as auxiliary. The corresponding final moments diagrams caused by the mentioned loads are shown in Figures 10.3, a, b.

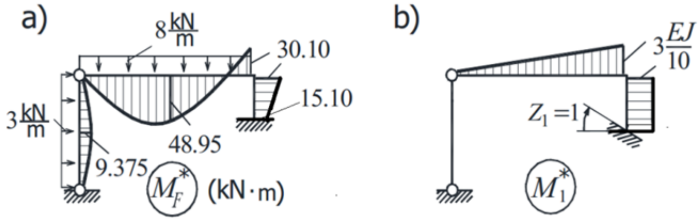


Figure 10.3

Further, following the well-known algorithm for calculating frames by the displacement method, we construct load diagram M_F (Figure 10.4, a), unit diagrams $\bar{M}_1$ and $\bar{M}_2$ (Figures 10.4, b, c) in the primary system. Ultimately, the final bending moments diagram (Figure 10.4, d) is constructed in the given system.

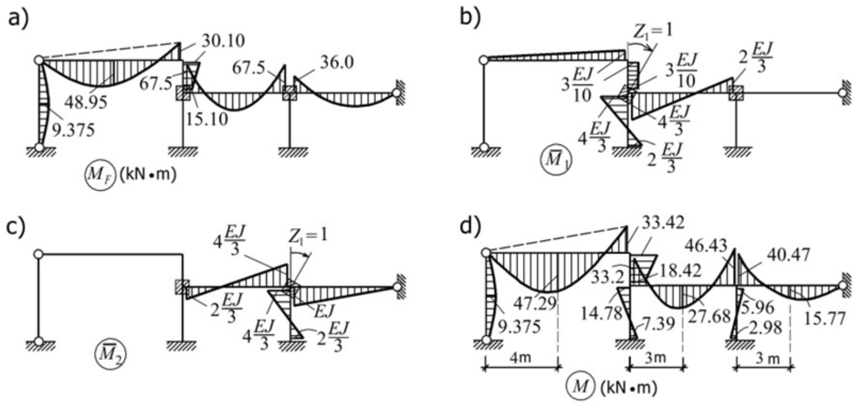


Figure 10.4

Let us consider another example. The frame (Figure 10.5, a) contains seven redundant constraints. However, we will calculate this system under the action of given load as a system containing three unknowns.

The primary system of the force method (this method is the main here) is shown in Figure 10.5, b. It includes statically indeterminate fragment

ABC and symmetric to it fragment $A'B'C'$. In order to build the moment diagrams in the primary system of the force method, first it is necessary to calculate these fragments due to the loads that they perceive.

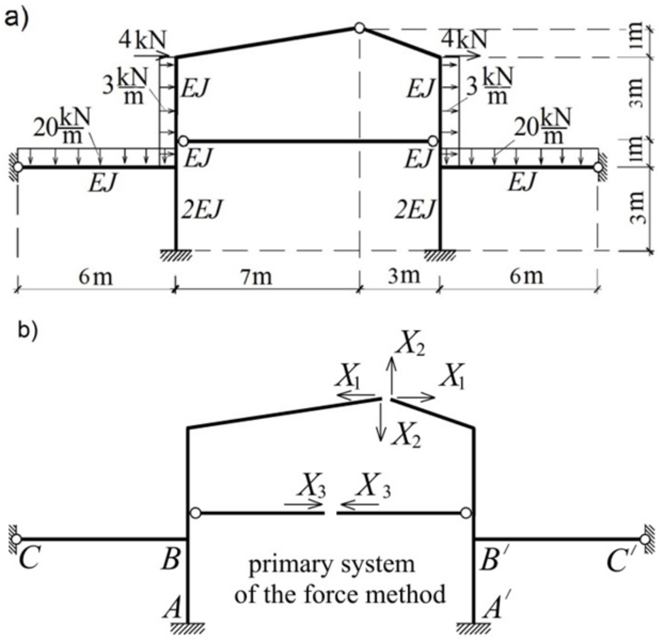


Figure 10.5

Frame ABC contains one unknown of the displacement method. The bending moment diagrams due to the action of a unit distributed load and a unit moment are shown in Figures 10.6, a, b.

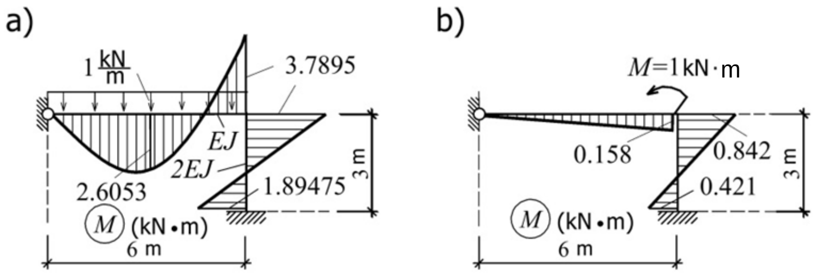


Figure 10.6

With their help, using the properties of linearly deformable systems, we construct a load diagram (Figure 10.7, a) and, as an example, the second unit diagram (Figure 10.7, b) of bending moments in the primary system of the force method.

Two other unit diagrams are constructed taking into account the distribution of moments on fragment ABC due to the $M = 1$ (Figure 10.6, b). The further calculation course corresponds to the algorithm of the force method.

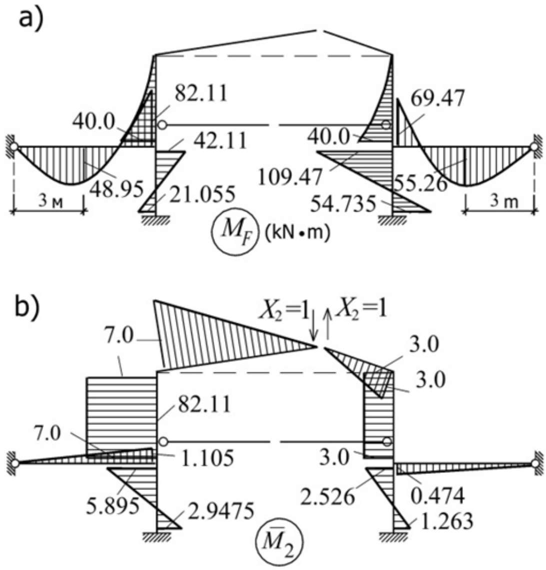


Figure 10.7

10.3. Mixed method

When calculating the frame by the mixed method, the main unknowns in one part are efforts in the redundant constraints. In the other, the remaining part, the unknowns are the displacements of the nodes. That is, during the calculation, both groups of unknowns (force method unknowns and displacement method unknowns) are determined simultaneously. The choice of unknowns, of course, is determined by the structure of the given frame. As a rule, in the part where a small number of redundant constraints are observed, the redundant links are removed and the primary unknowns of the force method are introduced, and in the other part, the additional constraints are introduced that prevent the angular and linear displacements of the nodes. These displacements are primary unknowns of the displacement method.

The system of equations from which these unknowns are determined is written on the basis of conditions similar to those used to write the canonical equations of the force method and the displacement method. We will give more complete explanations of the essentially mixed method by the example of calculation the frame shown in Figure 10.8, a.

Fragment *AD* of this frame (there is rigidly fixed support in node *D*) contains only two redundant constraints; it is convenient to calculate it by the force method; to calculate the rest part of the frame (its nodes are located at points *B*, *C*, *D*, *E*, *G*) it is more convenient to use the displacement method. Based on these considerations, we accept the primary system as it is shown in Figure 10.8, b.

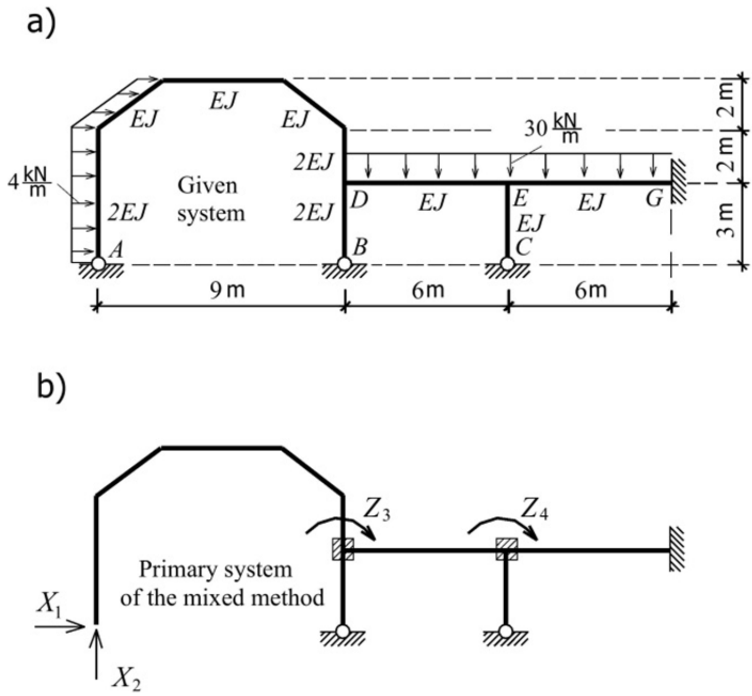


Figure 10.8

In this primary system the bending moment diagrams caused by the unit value of primary unknowns and by the given load are shown in Figure 10.9. From diagram $\bar{M}_1$ (Figure 10.9, a) it can be seen that force $X_1 = 1$ causes in the third additional link (its number corresponds to the number of the primary unknown) reactive force r'_{31} (note: the reaction (force) is caused by the force).

In the displacement method, the notation r_{31} would indicate the reaction in the third constraint caused by the displacement $Z_1 = 1$, that is, the causes of the reactions r'_{31} and r_{31} are different, therefore they are denoted differently. Similarly, the physical meaning of the reaction r'_{32} should be understood too.

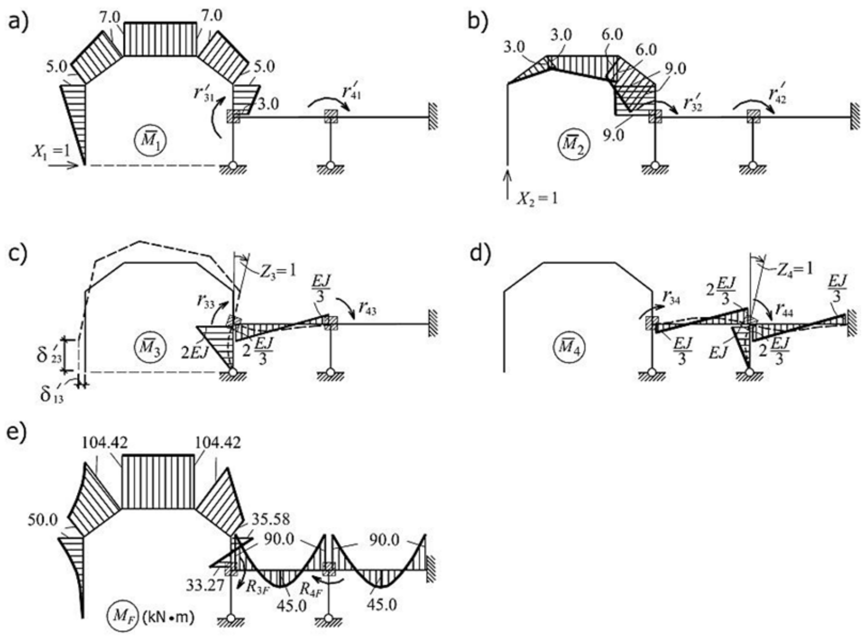


Figure 10.9

There is diagram $\bar{M}_3$ in Figure 10.9, c. The displacement of application point of force X_1 in its direction caused by displacement $Z_3 = 1$, is denoted by δ'_{13}. As in the case of the notation of reactions, writing δ'_{13} with the dash emphasizes the difference between this displacement and the displacement δ_{13}, caused by force $X_3 = 1$ (see the force method).

In accordance with the theorem of reciprocity of reactions and displacements (9.8) $r'_{31} = -\delta'_{13}$. Indeed, from the equilibrium equation of node D (Figure 10.9, a) it follows that $r'_{31} = 3{,}0$, Indeed, from the equilibrium equation of node D (Figure 10.9, a) it follows that displacement δ'_{13} occurs in the direction opposite to force $X_1 = 1$.

The value δ'_{13} can also be found by the rules for determining the displacements caused by the support settlements.

We now write down the canonical equations of the mixed method:

$$\left.\begin{aligned}
\delta_{11}X_1 + \delta_{12}X_2 + \delta'_{13}Z_3 + \delta'_{14}Z_4 + \Delta_{1F} &= 0,\\
\delta_{21}X_1 + \delta_{22}X_2 + \delta'_{23}Z_3 + \delta'_{24}Z_4 + \Delta_{2F} &= 0,\\
r'_{31}X_1 + r'_{32}X_2 + r_{33}Z_3 + r_{34}Z_4 + R_{3F} &= 0,\\
r'_{41}X_1 + r'_{42}X_2 + r_{43}Z_3 + r_{44}Z_4 + R_{4F} &= 0.
\end{aligned}\right\} \tag{10.1}$$

The first equation from this system expresses the condition that the displacement of the application point of force X_1 in its direction is equal to zero, where the first and second terms are the displacements caused by the forces X_1 and X_2, the third and fourth are displacements caused by the rotations of the nodes at angles Z_3 and Z_4, and the fifth is the displacement caused by the load. The meaning of the second equation is revealed in a similar way.

The third and the fourth equations have the meaning of the displacement method equations: the total reactions in the third and fourth additional constraints caused by the unit forces X_1, X_2 and the unit displacements Z_3, Z_4, as well as the load, are equal to zero.

In equations (10.1), the coefficients δ_{ik} and the free terms Δ_{iF} are determined in the same way as in the force method. For example,

$$\delta_{12} = \sum\int \frac{\bar{M}_1 \bar{M}_2\, dx}{EJ}, \qquad \Delta_{1F} = \sum\int \frac{\bar{M}_1 M_F\, dx}{EJ}.$$

Coefficients r_{ik} and free terms R_{iF} are determined by the ways used in the displacement method. For example, from the equilibrium of forces in the node D (Figure 10.9, d) we find $r_{34} = \dfrac{EJ}{3}$.

From the equation of equilibrium of moments in the same node on the bending moment diagram, caused by the given load, we obtain $R_{3F} = -123.27$.

The coefficients r'_{ik} and δ'_{ki}, as it has already been noted, are related by the ratio:

$$r'_{ik} = -\delta'_{ki}.$$

Analyzing the distribution of moments in Figures 10.9, a, b note that $r'_{41} = 0$ and $r'_{42} = 0$.

Having determined the coefficients and free terms, we obtain:

$$\left.\begin{array}{llllll}
\dfrac{446.17}{EJ}X_1 & -\dfrac{325.2}{EJ}X_2 & -3Z_3 & +0 & +\dfrac{5571.96}{EJ} & =0; \\
-\dfrac{325.2}{EJ}X_1 & +\dfrac{360.34}{EJ}X_2 & +9Z_3 & +0 & -\dfrac{3704.6}{EJ} & =0; \\
3X_1 & -9X_2 & +\dfrac{8}{3}EJ\cdot Z_3 & +\dfrac{1}{3}EJ\cdot Z_4 & -123.27 & =0; \\
0\cdot X_1 & +0\cdot X_2 & +\dfrac{1}{3}EJ\cdot Z_3 & +\dfrac{7}{3}EJ\cdot Z_4 & +0 & =0.
\end{array}\right\}$$

System solution:

$$X_1 = -16.165 \text{ kN}; \qquad X_2 = -5.476 \text{ kN};$$

$$Z_3 = 46.765\frac{1}{EJ} \text{ rad}; \qquad Z_4 = -6.681\frac{1}{EJ} \text{ rad}.$$

Final moment diagram is formed according to the following formula:

$$M = M_F + \bar{M}_1 X_1 + \bar{M}_2 X_2 + \bar{M}_3 Z_3 + \bar{M}_4 Z_4.$$

It is shown in Figure 10.10.

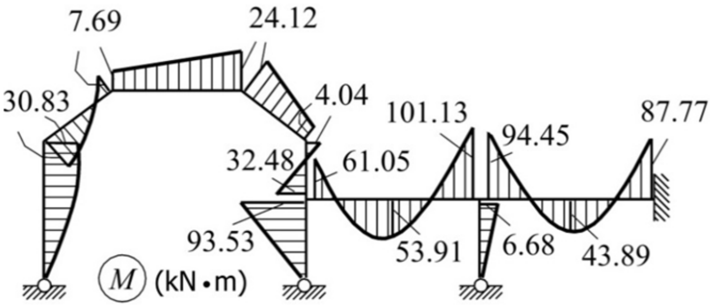


Figure 10.10

THEME 11. CALCULATING CONTINUOUS BEAMS

11.1. General information

A beam that overlaps two or more spans, not interrupted along its length by hinges is called a continuous beam.

The degree of static indeterminacy of continuous beams can be determined according to general rules (section 8.2). Since the beam is a single disk that overlaps several spans, the formula (8.2) is converted to:

$$\Lambda = L_0 - 3\,. \tag{11.1}$$

Beams shown in Figures 11.1, a, b contain two and three redundant constraints, respectively.

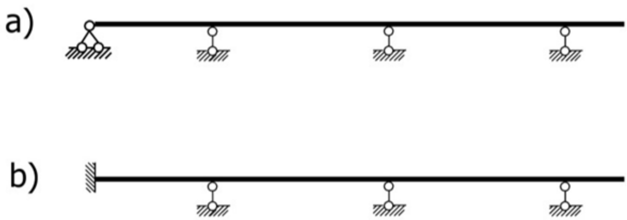


Figure 11.1

The reader is already familiar with the methods of calculation of statically indeterminate frames for various types of external actions (see the Themes 8, 9, 10). Features of the application of continuous beams calculation are considered in the next section.

11.2. Examples of Continuous Beam Calculation

E x a m p l e 1. Construct bending moments and shear forces diagrams for the continuous beam (Figure 11.2, a) using the force method.

The degree of static indeterminacy of the beam is equal to three. The primary system of the force method can be obtained by eliminating of support rods (Figure 11.3, b). Then, support reactions will be accepted as unknowns. It is easy to notice that in this case, none of the secondary coefficients of the canonical equations is equal to zero (Figure 11.3, c). The same can be seen, "multiplying" the corresponding unit moment diagrams. This means that such a primary system is not rational.

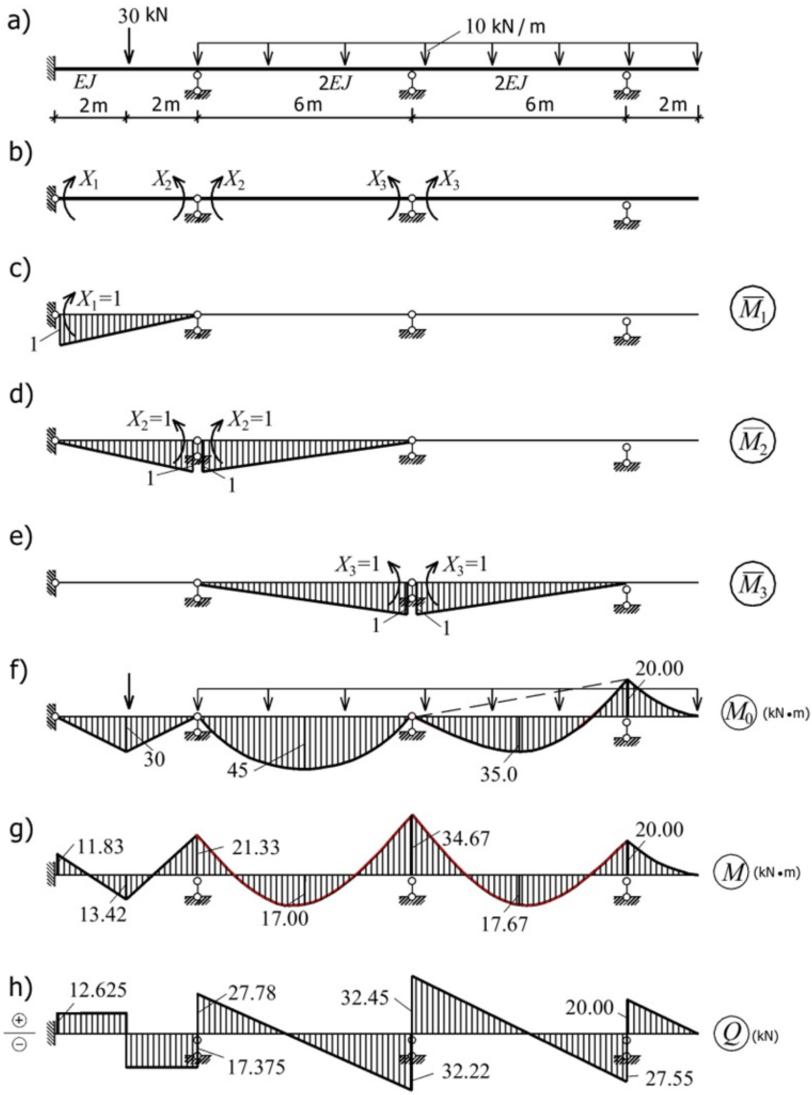


Figure 11.2

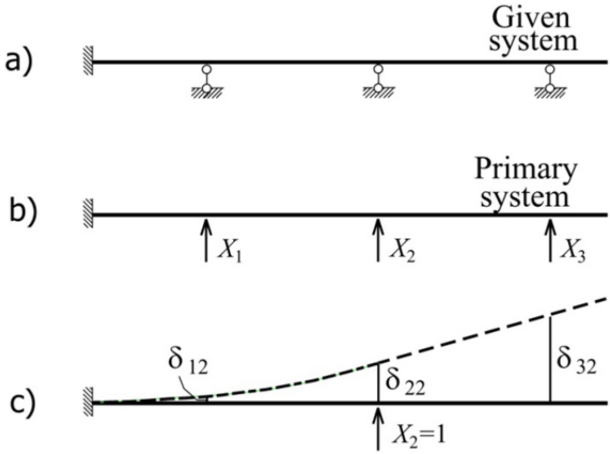


Figure 11.3

But the primary system obtained by introducing hinges into sections above the supports will be more successful (rational). (Figure 11.2, b). With this choice of the primary system, the continuous beam is divided into separate single-span beams. The primary unknowns in this case are the bending moments at the support cross-sections.

Having constructed in the primary system the bending moments diagrams caused by unit value of primary unknowns (Figures 11.2, c–d) and acting load (Figure 11.2, f), we calculate the coefficients and free terms of the canonical equations.

After simple transformations, we obtain the equations in the following form:

$$\left.\begin{aligned}\frac{4}{3}X_1+\frac{2}{3}X_2 \qquad\qquad +30=0;\\ \frac{2}{3}X_1+\frac{7}{3}X_2+\frac{1}{2}X_3+75=0;\\ \frac{1}{2}X_2+2X_3+80=0.\end{aligned}\right\}$$

We note that with the indicated way of selecting the primary system for a continuous beam, the first and last equations contein two unknowns, and all intermediate equations have three unknowns (the equation i contains unknowns X_{i-1}, X_i, X_{i+1}).

Having solved the system of equations, we find:

$$X_1 = -\frac{71}{6}\ \text{kN·m}, \qquad X_2 = -\frac{64}{3}\ \text{kN·m}, \qquad X_3 = -\frac{104}{3}\ \text{kN·m}.$$

The final bending moment diagram (Figure 11.2, g) is based on the expression:

$$M = M_F + \bar{M}_1 X_1 + \bar{M}_2 X_2 + \bar{M}_3 X_3 .$$

The analytical expression for determining the bending moment in the cross-section, which is located between the support points of the beam, can be obtained from formula (8.16).

A kinematic check of the correctness of the diagram M consists in checking the displacements in the directions of the primary unknowns and is performed according to the formula (8.23).

The shear forces diagram is shown in Figure 11.2, h.

To determine the reaction in the support with the number n (Figure 11.4), we cut out an infinitely small section of the beam using two sections located on both sides of the support, and show the shear forces in these cross-sections. From the equation $\sum Y = 0$ it follows that:

$$R_n = Q_{n+1} - Q_n .$$

In particular, in the fixed support (Figure 11.5) and the first intermediate support (Figure 11.6), the vertical reactions are 12.625 kN and 45.155 kN, respectively (Figure 11.6).

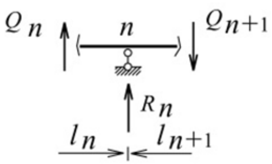


Figure 11.4

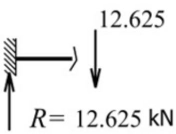


Figure 11.5

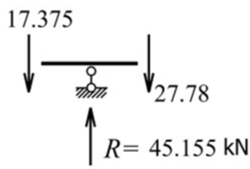


Figure 11.6

E x a m p l e 2. Calculate the same beam (Figure 11.7, a) using the displacement method.

The degree of kinematic indeterminacy of a continuous beam is a variable characteristic. Indeed, any cross-section of the beam can be declared as a node in which two rods are joined. Such a node, in the general case, will have two degrees of freedom: vertical displacement and the angle of rotation (displacement along the beam axis according to the accepted assumptions for a linearly deformable system is not taken into account). In the primary system of the displacement method, such a node must be fixed with two additional constraints. As a result, the dimension of the beam calculating problem increases.

In order to reduce the dimension of the problem, it is advisable to consider only support nodes. Each section above the supports of the beam has only one degree of freedom – the angle of rotation.

For the given beam (Figure 11.7, a), we choose the primary system of the displacement method shown in Figure 11.7 a. The primary unknowns are the angles of rotation of the support sections of the beam.

In Figures 11.7, b–d, the bending moment diagrams caused by unit value of primary unknowns are shown, and in Figure 11.7, d the bending moment diagram caused by the acting load is shown.

Having calculated the coefficients and free terms of the canonical equations according to well-known rules, we obtain a system of equations in the following form:

$$\left.\begin{aligned} &\frac{7}{3}EJ\,Z_1+\frac{2}{3}EJ\,Z_2 -15=0;\\ &\frac{2}{3}EJ\,Z_1+\frac{8}{3}EJ\,Z_2+\frac{2}{3}EJ\,Z_3=0;\\ &\frac{2}{3}EJ\,Z_2+\frac{4}{3}EJ\,Z_3+10=0. \end{aligned}\right\} \tag{11.2}$$

Having solved the equations system (11.2), we find:

$$Z_1=\frac{19}{3EJ}\ \text{rad},\qquad Z_2=\frac{1}{3EJ}\ \text{rad},\qquad Z_3=-\frac{23}{3EJ}\ \text{rad}.$$

Final bending moment diagram is constructed from the expression:

$$M = M_F + \bar{M}_1 Z_1 + \bar{M}_2 Z_2 + \bar{M}_3 Z_3 ,$$

It has the form shown in Figure 11.7, g.

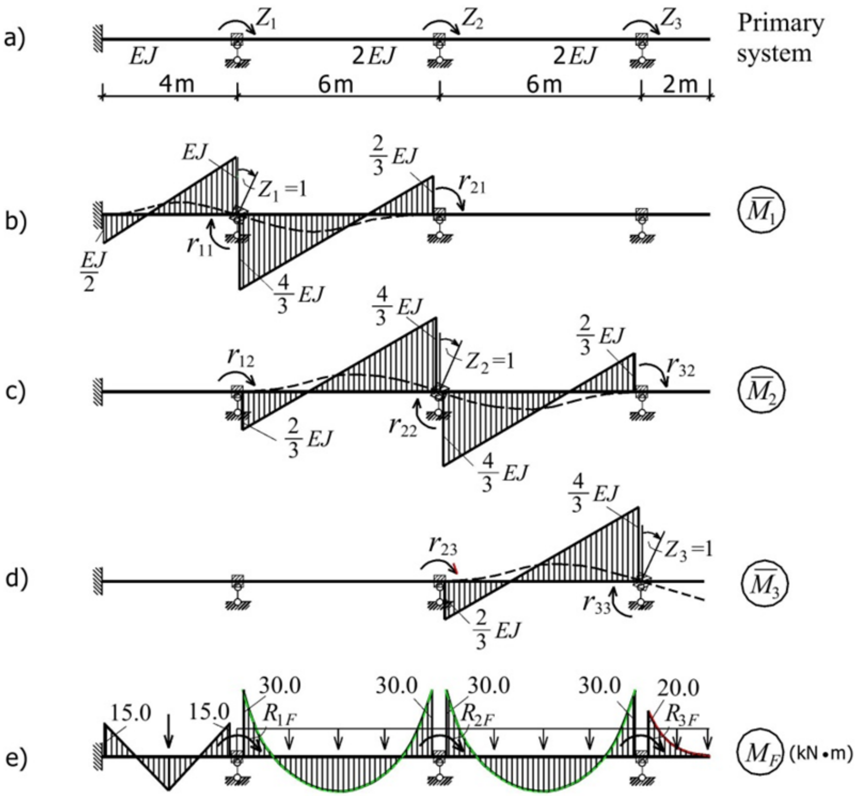


Figure 11.7

The same beam (Figure 11.7, a) also can be calculated using the displacement method as a beam with two unknowns. In this case, the primary system (Figure 11.8, a) include the “non-standard” element shown in Figure 11.9, a (it is not in the set of elements in table. 9.1). The bending moment diagrams caused by unit value of two primary unknowns are shown in Figures 11.8, b, c.

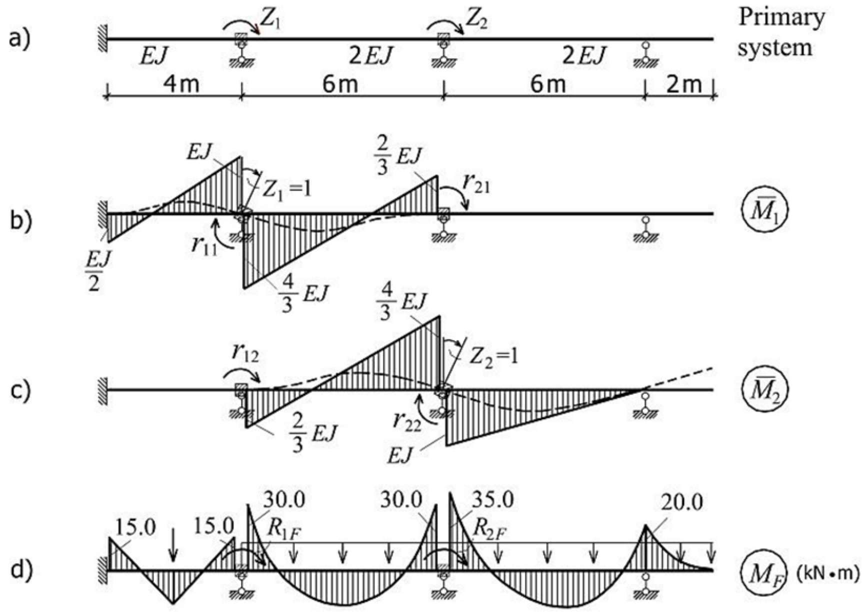


Figure 11.8

The calculation of the “non-standard” element (Figure 11.9, a) on the action of a uniformly distributed load is performed by the force method. This load moment diagram (Figure 11.9, b) is used to construct the total bending moments diagram M_F due to the given load (Figure 11.8, d) in the primary system with two primary unknowns

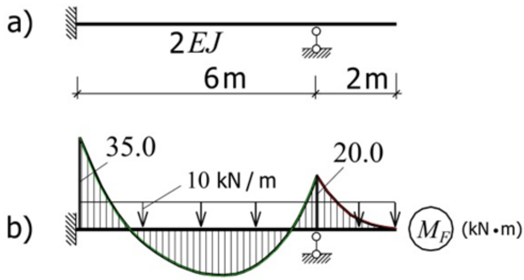


Figure 11.9

The canonical equations, after determining the coefficients and free terms, are written in the form:

$$\left.\begin{aligned} \frac{7}{3}EJ\,Z_1+\frac{2}{3}EJ\,Z_2-15=0; \\ \frac{2}{3}EJ\,Z_1+\frac{7}{3}EJ\,Z_2-\;5=0. \end{aligned}\right\} \qquad (11.3)$$

Solving them, we get:

$$Z_1=\frac{19}{3EJ}\ \text{rad}, \qquad Z_2=\frac{1}{3EJ}\ \text{rad}.$$

Naturally, the final moment diagram will be the same as in Figure 11.7, g.

Note the following. Removing the additional link in the primary system (Figure 11.7, a) allowed us to move from the system of equations (11.2) to the system (11.3). This transition could be carried out without calculating the beam as a twice kinematically indeterminate system.

To solve the system of equations (11.2), we apply Jordan eliminations (Gauss method). The coefficients and free terms of system (11.2) are written in the form of table 11.1 (the factors EJ in front of the unknowns Z_i are not included in the table) and we take one step of ordinary Jordanian eliminations, taking the coefficient r_{33} as the resolving element.

Table 11.1

			↑		
		Z_1	Z_2	Z_3	1
	0 =	$\frac{7}{3}$	$\frac{2}{3}$	0	–15
	0 =	$\frac{2}{3}$	$\frac{8}{3}$	$\frac{2}{3}$	0
←	0 =	0	$\frac{2}{3}$	$\boxed{\frac{4}{3}}$	10

Table 11.2

	Z_1	Z_2	0	1
0 =	$\frac{7}{3}$	$\frac{2}{3}$	0	–15
0 =	$\frac{2}{3}$	$\frac{7}{3}$	$\frac{1}{2}$	–5
Z_3 =	0	$-\frac{1}{2}$	$\frac{3}{4}$	$-\frac{15}{2}$

Transition from Table 11.1 to table 11.2 is carried out according to the following rules.

1. The resolving element $\left(a_{rs}=\frac{4}{3}\right)$ is replaced by the inverse value.

2. The remaining elements of the resolving column (s) are divided by the resolving element.

3. The remaining elements of the resolving row (r) are divided by the resolving element and change signs.

4. Other elements are calculated by the formula

$$b_{ij}=\frac{a_{ij}a_{rs}-a_{is}a_{rj}}{a_{rs}},$$

at $i\neq r,\quad j\neq s$ (according to the rule of the rectangle).

In Table 11.2, the coefficients and free terms of the system of equations (11.3) are written. Zero columns could not be written to the table. From this table it follows that:

$$EJ\,Z_3=-\frac{1}{2}EJ\,Z_2-\frac{15}{2}.$$

E x a m p l e 3. It shows the calculation of the beam (Figure 11.2, a) by the mixed method.

There are many variations of the primary systems of the mixed method. Some of them are shown in Figure 11.10. To demonstrate the features of the mixed method, we choose the primary system shown in Figure 11.11, a.

The bending moment diagrams caused by unit value of primary unknowns and acting load are shown. in Figures 11.11, b–e.

The system of canonical equations for the accepted primary unknowns has the following form:

$$\left.\begin{aligned}\delta_{11}X_1+\delta_{12}X_2+\delta'_{13}Z_3+\Delta_{1F}&=0;\\ \delta_{21}X_1+\delta_{22}X_2+\delta'_{23}Z_3+\Delta_{2F}&=0;\\ r'_{31}\,X_1+r'_{32}\,X_2+r_{33}\,Z_3+R_{3F}&=0.\end{aligned}\right\}$$

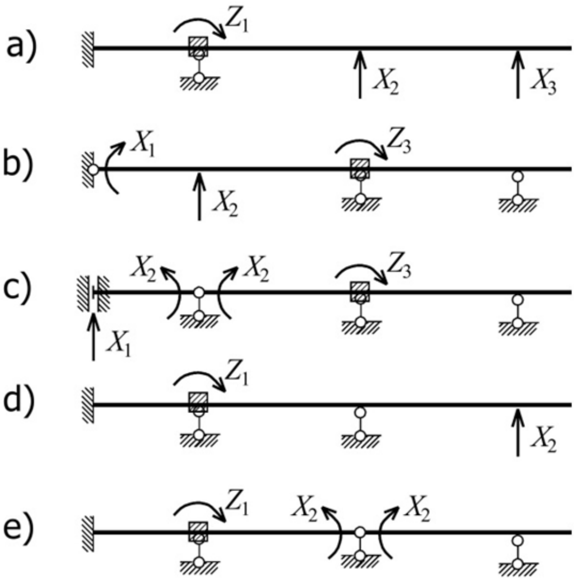


Figure 11.10

We will determine the free terms of the first and second equations, as in the force method by “multiplying” the moment diagrams:

$$\Delta_{1F} = \sum\int \frac{\bar{M}_1 M_F \, dx}{EJ} = \frac{1}{EJ}\frac{1}{2}30 \cdot 4 \cdot 0.5 = \frac{30}{EJ};$$

$$\Delta_{2F} = \sum\int \frac{\bar{M}_2 M_F \, dx}{EJ} = \frac{30}{EJ} + \frac{6}{6 \cdot 2EJ}\left(4 \cdot 0.25 \cdot 22.5 + 45 \cdot 0.5\right) = \frac{52.5}{EJ}.$$

The free term R_{3F} is determined from the equation of equilibrium of moments in the node with additional fixed link: $R_{3F} = 10.0$.

Since $r'_{ik} = -\delta'_{ki}$, then $r'_{32} = 0.5$, and $\delta'_{23} = -0.5$.

The determination of other coefficients at unknowns is carried out according to the rules set out in chapters 8, 9.

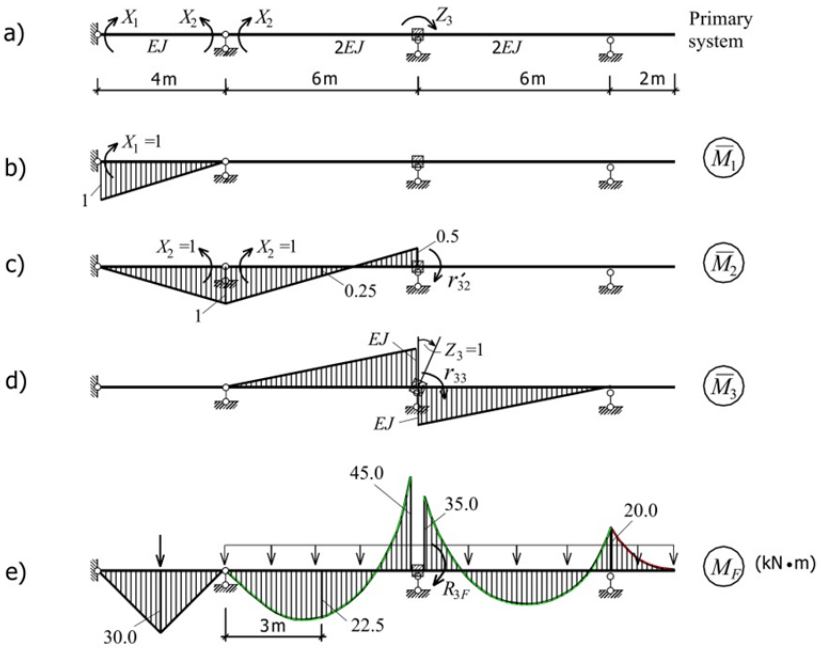


Figure 11.11

Numerically the canonical equations have the following form:

$$\left.\begin{aligned}
&\frac{4}{3EJ}X_1+\frac{2}{3EJ}X_2+\frac{30}{EJ}=0;\\
&\frac{2}{3EJ}X_1+\frac{12.5}{6EJ}X_2-0.5Z_3+\frac{52.5}{EJ}=0;\\
&0.5\,X_2+2\,EJ\,Z_3+10=0.
\end{aligned}\right\}$$

Having solved the system, we obtain:

$$X_1=-11.83\,\text{kN m};\qquad X_2=-21.33\ \text{kN}\cdot\text{m};\qquad Z_3=\frac{1}{3EJ}\ \text{rad.}$$

Final bending moment diagram is constructed from the expression:

$$M=M_F+\overline{M}_1X_1+\overline{M}_2X_2+\overline{M}_3Z_3.$$

It has the same form as in Figure 11.2 g.

E x a m p l e 4. The displacements of the supports of the continuous beam are shown in Figure 11.12, a. It is necessary to construct the bending moment diagram, taking $c_1 = 0.01$ rad, $c_2 = c_3 = 0.06$ m.

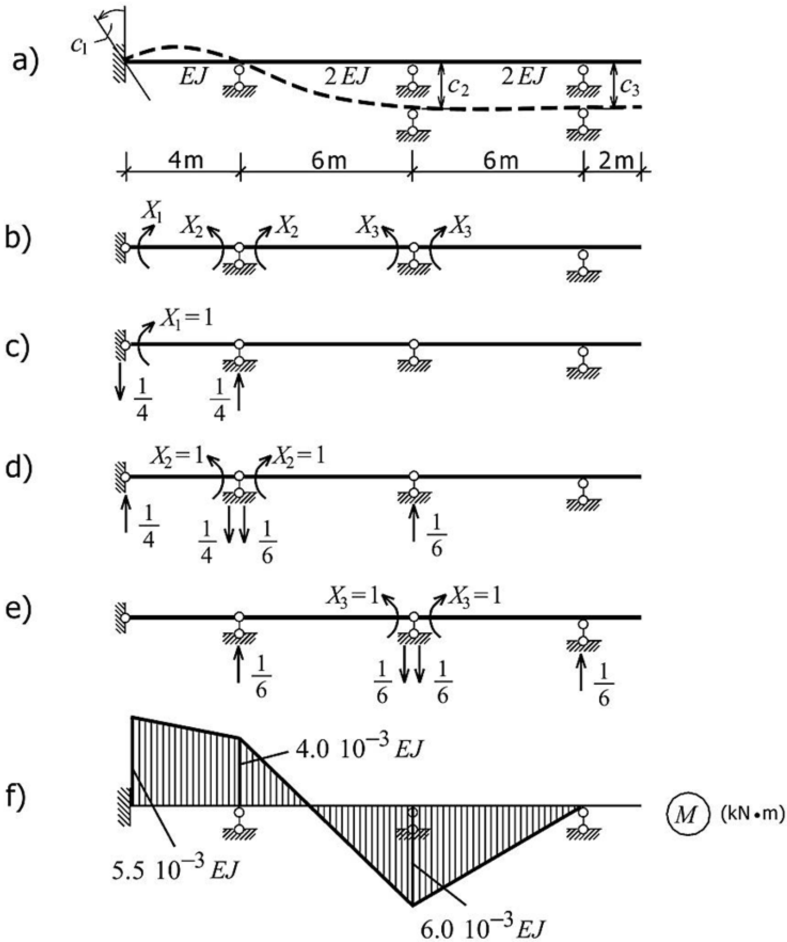


Figure 11.12

Calculation of the beam subjected by the displacements of the supports is carried out by the methods considered earlier. We show the solution by the force method.

Let the primary system be the same as shown in Figure 11.12, b. The free terms of the canonical equations are determined by the formula (7.13). Using the distribution of reactions in the supports (Figures 11.12, c–d), we find:

$$\Delta_{1c} = -\sum R_{\kappa 1} c_\kappa = -\left(-1 \cdot c_1\right) = c_1 = 0.01;$$

$$\Delta_{2c} = -\sum R_{\kappa 2} c_\kappa = -\left(-\frac{1}{6} c_2\right) = \frac{1}{6} c_2 = 0.01;$$

$$\Delta_{3c} = -\sum R_{\kappa 3} c_\kappa = -\left(\frac{1}{3} c_2 - \frac{1}{6} c_3\right) = -\frac{1}{3} c_2 + \frac{1}{6} c_3 = -0.01.$$

The coefficients at unknowns have the same values as in example 1.

We write the canonical equations corresponding to the given displacements of supports:

$$\left.\begin{aligned} \frac{4}{3} X_1 + \frac{2}{3} X_2 \qquad\qquad & + 0.01\, EJ = 0; \\ \frac{2}{3} X_1 + \frac{7}{3} X_2 + \frac{1}{2} X_3 & + 0.01\, EJ = 0; \\ \frac{1}{2} X_2 + 2\, X_3 & - 0.01\, EJ = 0. \end{aligned}\right\}$$

Solving them, we get:

$$X_1 = -5.5 \cdot 10^{-3} EJ \ \text{kN·m}; \qquad X_2 = -4.0 \cdot 10^{-3} EJ \ \text{kN·m};$$

$$X_3 = 6.0 \cdot 10^{-3} EJ \ \text{kN·m}.$$

The diagram M is shown in Figure 11.12, f.

11.3. Constructing Influence Lines for Internal Forces

To construct influence lines for internal forces by the static method (see Sections 8.11, 9.11), it is necessary, in the general case, to calculate a continuous beam on the action of force $F = 1$, applied at a number of characteristic points of each span and compose an influence matrix L_S of internal forces. Based on the values of the i-th row elements, you can construct an influence line S_i.

We calculate the beam on action of the unit force in the sections indicated in Figure 11.13, a and, based on the calculation results, construct the influence lines for internal forces.

The shape of the influence lines for each span, as a rule, is determined by the values of its ordinates in three intermediate points. For example, to build influence line M_c (Figure 11.13, b) or influence line M_5 (Figure 11.13, c) it is enough to find the corresponding bending moments M_c or M_5 when the force $F=1$ locates in the cross-sections dividing the span into four parts.

The shape of influence line for bending moments can have some features when it is constructed for sections located near the supports. So, when constructing the influence line for bending moment for the section K_2 (Figure 11.13, d), it turns out that the force $F=1$ located to the right-hand of the second span does not cause a bending moment in this cross-section (such point K_2 is called the left focus of the second span).

If a certain cross-section K_1 is located between the points B and K_2, then the influence lines for the bending moment in this cross-section of this span will be have double-sign (Figure 11.13, e). Therefore, in order to avoid errors, in the part of the span to which the effort under investigation belongs, the number of trial installations of force $F=1$ should be taken as increased.

It is known that the shape of the influence line, in accordance with the kinematic method (see Section 8.11), is similar to the diagram of beam deflections caused by the displacement of the corresponding link in its direction by value equal to one. For example, to get the outline of inf.line Q_5 (Figure 11.13, f) it is necessary to move apart the ends of the beam adjacent to the fifth cross-section vertically by a length equal to one so that this ends remain parallel to each other.

The numerical values of the ordinates of the influence line are conveniently calculated by the static method.

To establish the form of Influence Line V_B (Figure 11.13, g) it is necessary to remove the support rod at the point B in the beam and give in its direction the displacement equal to unit. The outline of the curved axis of the beam will correspond to the outline of the required influence line. The ordinates of the influence line are determined by the static method.

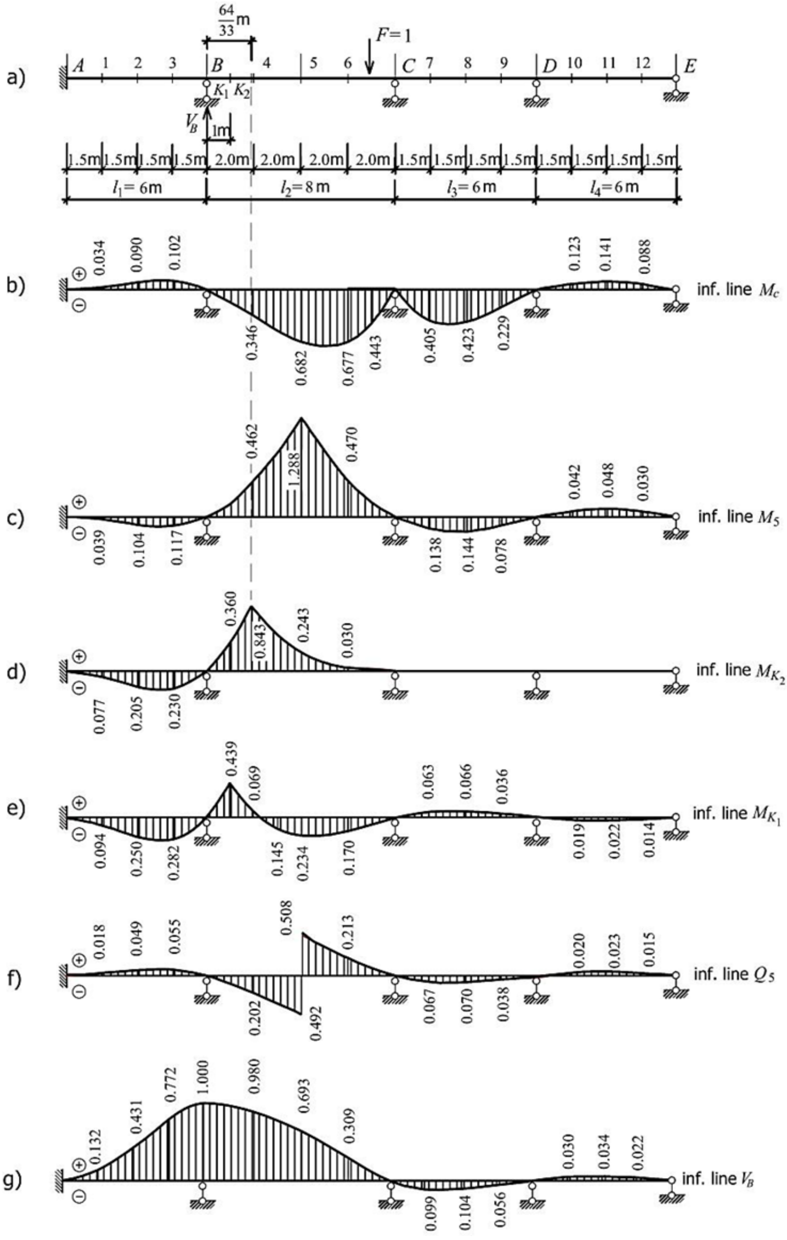

a)
b)
c)
d)
e)
f)
g)
inf. line M_c
inf. line M_5
inf. line M_{K_2}
inf. line M_{K_1}
inf. line Q_5
inf. line V_B

Figure 11.13

11.4. Enveloping Diagrams of the Internal Forces

Continuous beams, like most other structures, are loaded with both constant and temporary loads, the nature of the action of which, in the general case, turns out to be rather arbitrary: it can be in all spans of the beam or only in some of them. Extremal efforts in beam cross-sections are dependent on unfavorable loadings the location of which is determined by using influence lines (see chapter 3).

However, this method of finding extremal efforts is quite complicated and, moreover, does not provide a clear idea of the distribution of the maximum and minimum efforts along the length of the beam.

The problem of determining extremal efforts is solved more simply using enveloping diagrams of internal forces. Consider the problem of constructing enveloping diagrams of bending moments in a continuous beam loaded with a constant (Figure 11.14, a) and temporary loads (Figure 11.14, b). The moment diagram due to the constant load is shown in Figure 11.14, c. The moment diagrams due to the serial loading of each span with a temporary load are shown in Figures 11.14, d–g.

The maximum and minimum bending moments in the beam cross-sections are determined by the expressions:

$$\max M = M_{const} + \sum M^{+}_{temp}\,; \qquad \min M = M_{const} + \sum M^{-}_{temp}\,,$$

where M_{const} is the bending moment in the given cross-section due to the constant load;

M^{+}_{temp} is a positive bending moment in the given cross-section due to temporary loads in a corresponding span;

M^{-}_{temp} is a negative bending moment in the given cross-section due to temporary loads in a corresponding span.

For example, $\max M_7 = 34.03 + 33.63 + 0.90 = 68.56$ kN·m;

$\max M_{10} = -44.03 + 2.26 + 2.64 = -39.13$ kN·m;

$\min M_1 = -21.04 - 24.39 - 1.02 = -46.45$ kN·m;

$\min M_{11} = 5.99 - 10.9 - 4.28 = -9.19$ kN·m.

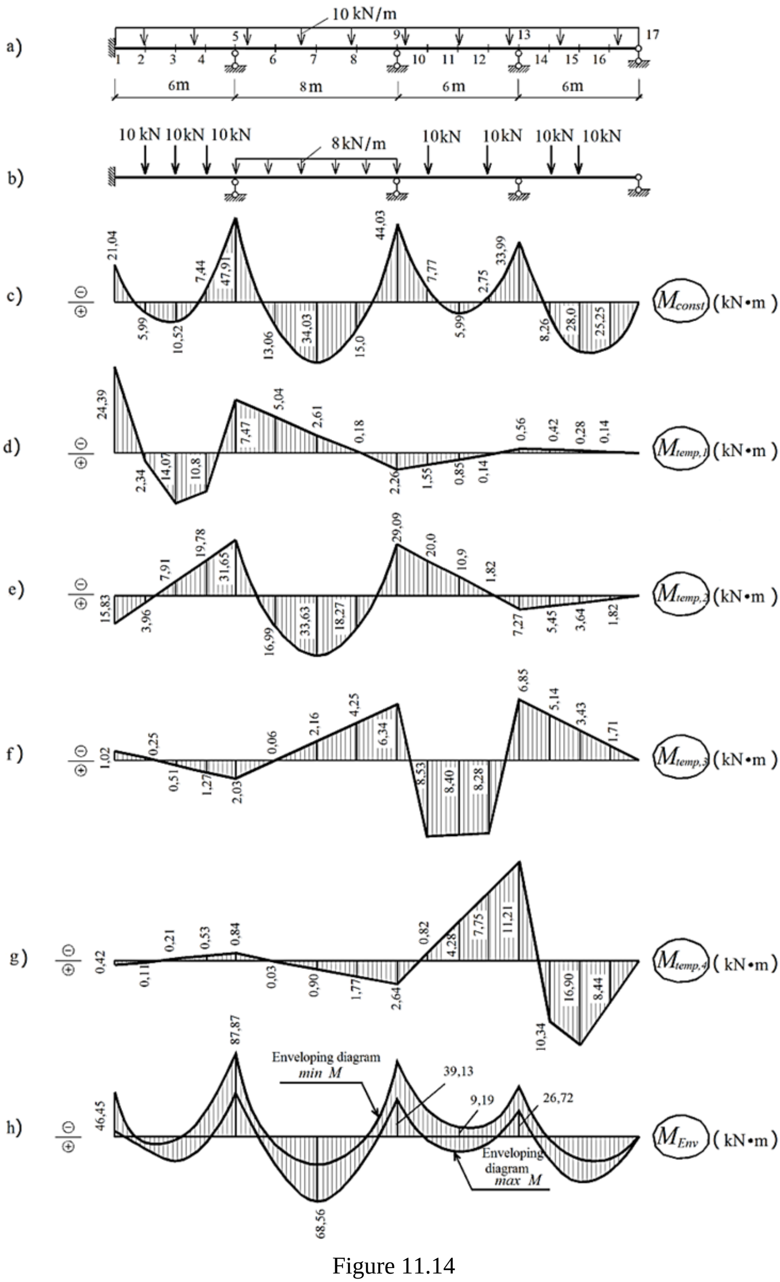


Figure 11.14

Connecting by the smooth curve the points corresponding to the values of $\max M$, we obtain the enveloping diagram of the maximum moments (Figure 11.14, h). The enveloping diagram of the minimum moments corresponds to the values of $\min M$.

From the constructed graphs it follows that in some parts of the beam the stretched fibers of the beam are located only below (or only at the top), and in other parts, the stretched fibers can be located both below and above. In the cross-section 11 $\max M_{11} = 15.24$ kN·m (not shown in Figure 11.14), and $\min M_{11} = -9.19$ kN·m.

Information about the distribution of calculated efforts is used in the design of beams.

A similar approach to the construction of enveloping diagrams of bending moments, shear and longitudinal forces can be applied in the calculation of other structures.

11.5. Calculating Continuous Beams on Elastic Supports

Examples of elastic supports are long columns, on which a continuous beam rests (Figure 11.15, a), transverse beams of the carriageway of a metal bridge, on which longitudinal continuous beams rest, as well as pontoons, which serve as supports of the floating bridge.

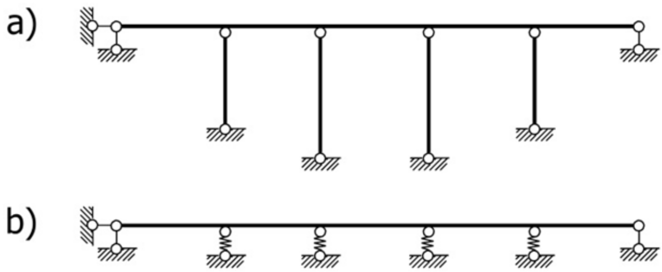


Figure 11.15

In the design scheme of the beam, such supports are depicted in the form of springs (Figure 11.15, b). If the elastic supports are linearly deformable, then the displacements of the support points of the beam are proportional to the reactions of the supports:

$$y_m = c_m \ R_m \, ,$$

where c_m – is the pliability coefficient of the m-th support, m/kN.

Calculation of continuous beams on elastic supports is conveniently performed by the force method. The primary system of the force method is accepted the same as in the calculation of beams on non-deformable (absolutely rigid) support rods. A fragment of the primary system of a multi-span continuous beam is shown in Figure 11.16, a. Emphasizing the physical meaning of the primary unknowns of the force method, in practical calculations the notation X_i is replaced with M_i .

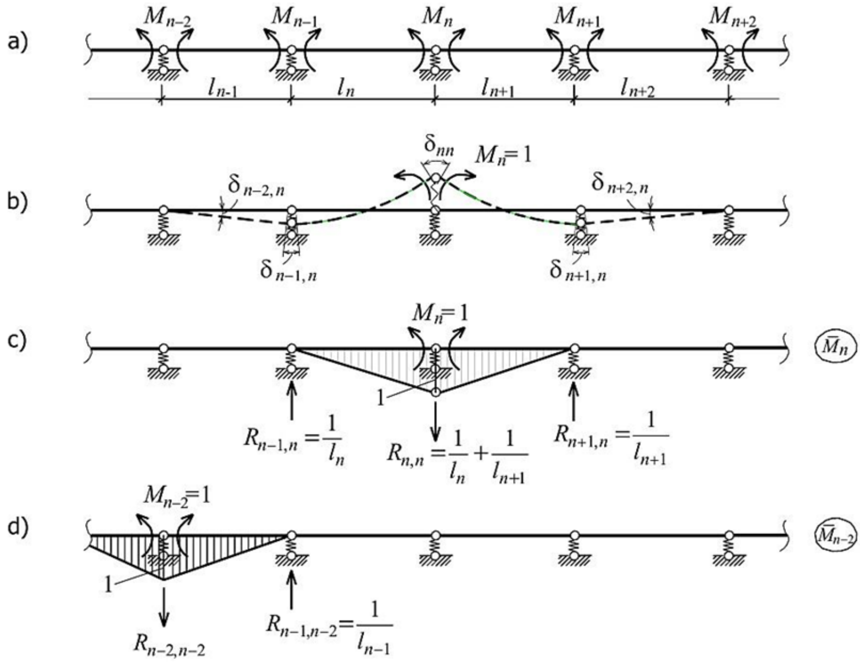


Figure 11.16

Displacement in the direction of the unknown M_n (the angle of mutual rotation of the cross-sections of the beams adjacent to the n-th support) will be caused only by the support moments M_{n-2}, M_{n-1}, M_n,

M_{n+1}, M_{n+2} and the load located in the spans l_{n-1}, l_n, l_{n+1}, l_{n+2}, therefore, the corresponding canonical equation of the force method has the form:

$$\delta_{n,n-2}M_{n-2} + \delta_{n,n-1}M_{n-1} + \delta_{nn}M_n +$$
$$+\delta_{n,n+1}M_{n+1} + \delta_{n,n+2}M_{n+2} + \Delta_{nF} = 0.$$

It is called an equation of five moments.

The deformed state of the primary system caused by $M_n = 1$ is shown in Figure 11.16, b. In Figures 11.16, c the moments diagram and the values of the support reactions due to $M_n = 1$ are presented. The moments diagram and the reactions due to $M_{n-2} = 1$ are given in Figures 11.16, d.

The coefficients and free terms of the equations are determined by the Mohr-Maxwell formula, taking into account the influence of bending moments in the beam and reactions in elastic supports:

$$\delta_{ik} = \sum\int \bar{M}_i \frac{\bar{M}_k dx}{EJ} + \sum c_m R_{mi} R_{mk} ,$$

$$\Delta_{iF} = \sum\int \bar{M}_i \frac{M_F dx}{EJ} + \sum c_m R_{mi} R_{mF} ,$$

where $\bar{M}_i, \bar{M}_k$ are the moments in the beam, respectively, due to $M_i = 1$ and $M_k = 1$;

M_F are the moments in the beam due to given load;

R_{mi}, R_{mk} are the reactions in the support m, respectively, due to $M_i = 1$ and $M_k = 1$;

R_{mF} is the reaction in the support m due to given load;

c_m – is the pliability coefficient of the m-th support.

Influence lines for internal forces in beams on elastic supports, as in beams on absolutely rigid supports, are constructed by static and kinematic methods.

THEME 12. CALCULATING STATICALLY INDETERMINATE TRUSSES

12.1. Types of Statically Indeterminate Trusses

This chapter discusses the features of calculating trusses as articulated-rod systems with extra connections (with redundant links). It is essential to remember that the nodal joints of the hinge-rod systems are ideal hinges without friction.

The degree of static indeterminacy of the hinge-rod system is determined by the formula

$$\Lambda = B + L - 2N,$$

where B is the number of rods making up the truss;
L – the number of support rods of the truss;
N – the number of truss nodes.

Examples of several types of statically indeterminate trusses are shown below (Figures 12.1. and 12.2, a).

A three-span continuous beam truss with parallel chords and a triangular lattice (Figure 12.1, a) is twice externally statically indeterminate. After being detached from the supports, this truss has a geometrically unchangeable statically determinate structure.

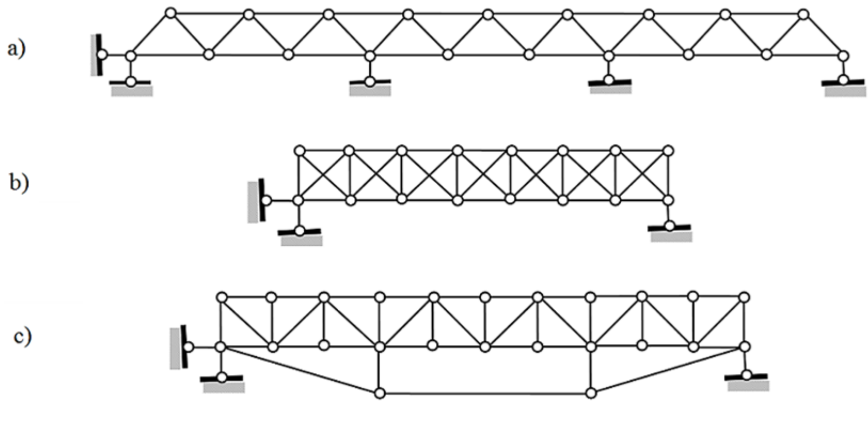


Figure 12.1

The seven-panel beam truss with a crossed lattice (Figure 12.1, b) contains seven extra links. This truss is internally statically indeterminate. Externally, it is statically determinate: the reactions of its supports can be found from the equilibrium equations, as in a simple beam.

A beam truss with parallel chords, with a triangular lattice and additional struts, but strengthened with a polygonal tie (Figure 12.1, c), is also internally statically indeterminate once.

A thrusting two-hinged truss with an additional brace in the central panel (Figure 12.2, a) is statically indeterminable both externally and internally.

12.2. Features of Calculating Statically Indeterminate Trusses

The calculation of statically indeterminate trusses is performed, as a rule, by the force method. The primary system of the force method is selected by cutting the truss rods, or by removing the support links (Figure 12.2, b), which are not absolutely necessary.

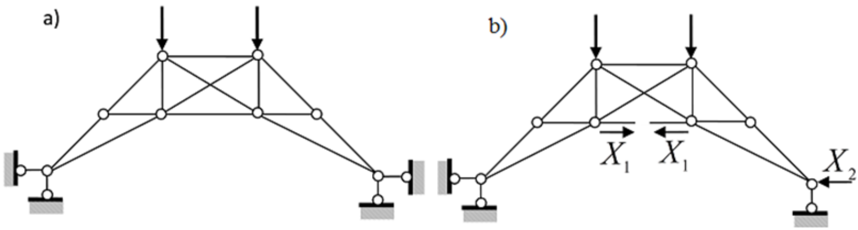


Figure 12.2

The canonical equations of the force method have standard form

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \dots & \delta_{1n} \\ \delta_{21} & \delta_{22} & & \delta_{2n} \\ \dots & \dots & \dots & \dots \\ \delta_{n1} & \delta_{n2} & \dots & \delta_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \dots \\ X_n \end{bmatrix} + \begin{bmatrix} \Delta_{1F} \\ \Delta_{2F} \\ \dots \\ \Delta_{nF} \end{bmatrix} = 0,$$

where the index n means the number of primary unknowns of the forces method.

With a nodal load in the rods of statically indeterminate trusses, as well as other hinged-rod systems, only longitudinal internal forces will arise. Therefore, the displacements in the trusses will depend only on the longitudinal deformations of their rods, and the one-term Maxwell formula should be used to calculate the displacements.

Consequently

$$\delta_{ik} = \Sigma \frac{sN_i N_k}{EA}; \quad \Delta_{iF} = \Sigma \frac{sN_i N_F}{EA} \quad (i,k = 1, 2,..., n)$$

where the summation sign $\sum$ extends to all the truss rods;

N_i, N_k, N_F – accordingly, the efforts in the rods of the primary system of the method of forces due to unit values of the primary unknowns $(X_i = 1,\ X_k = 1)$ and the given load F;

s and EA – length and tensile-compression rigidity of the corresponding truss rod.

The final effort in the rods of statically indeterminate truss is calculated by the formula

$$N = N_F + \sum_{i=1}^{n} N_i X_i .$$

All calculations are conveniently carried out in a tabular form. For a truss with two primary unknowns (Figure 12.2), such a table can have the following form (Table 12.1).

Table 12.1

№№ rods	$b=s/(EA)$	N_F	N_1	N_2	$N_1 N_1 b$	$N_1 N_2 b$	$N_2 N_2 b$	$N_1 N_F b$	$N_2 N_F b$	$N_1 X_1$	$N_2 X_2$	N
1	2	3	4	5	6	7	8	9	12	11	12	13
1												
$\sum$					δ_{11}	δ_{12}	δ_{22}	Δ_{1F}	Δ_{2F}			

The first column of the table shows the rod numbers in a selected order. The second column contains the deformability of the rods, i.e. the ratio of the lengths of the rods to their longitudinal rigidity. The third, fourth and fifth columns contain the internal forces in the truss rods, calculated in the primary system of the force method due to the given load and unit values of the primary unknowns.

In the next five columns, the actual calculations are performed, the meanings of which are indicated in the header of the table. The sums of the elements of the columns 6,...,10 give the values of the coefficients at primary unknowns and the values of the free terms of the canonical equations of the force method, i. e. the displacements from unit unknowns and the displacements from given load in the primary system.

After the values of the basic unknowns are determined from the solution of the system of canonical equations, columns 11 and 12 are filled in. In other words, the actual values of the efforts in the primary system are calculated by the found values of the primary unknowns.

Finally, summing columns 3, 11 and 12, the final values of the internal forces in the rods of a statically indeterminate truss are obtained. If necessary, additional columns can be added to the table for intermediate and final kinematic checks in accordance with the force method.

Calculation of trusses by the displacement method leads to a significantly larger number of primary unknowns. As a rule, the displacement method is used in the automated calculation of trusses using computers.

12.3. Constructing Influence Lines for Efforts in Truss Rods

The influence lines are used to calculate trusses under the action of a moving load to determine its most unfavorable location. Based on the theorem on the reciprocity of reactions and displacements (the kinematic method of constructing influence lines), the influence line for the internal force in any rod (link) of a statically indeterminate truss coincides with the deflection line of the loaded chord of the truss caused by the action of a unit displacement in the direction of this internal force (in the direction of the corresponding link).

The process of constructing the influence line for the effort in a certain rod (link) of a statically indeterminate truss can be carried out somewhat differently, based on the theorem on reciprocity of displacements. To build the influence line for the effort in a truss rod, it is necessary to cut this rod (remove the corresponding link). The degree of static indeterminacy of the truss is reduced by one. A truss with a removed link can be considered as the primary system of the force method, in the general case, statically indeterminate. The primary unknown, the reaction in the removed constraint, depends on the point of application of the mobile force equaled to one. The law of change of this primary unknown determines the desired line of influence.

From the corresponding canonical equation it may be found that:

$$Inf.Line\ of\ X_1 = \frac{-\delta_{1F}(x)}{\delta_{11}} = \frac{-\delta_{F1}(x)}{\delta_{11}}$$

where δ_{11} is the displacement in the primary system in the direction of the removed constraint from the unit value of the force in this constraint, that is constant quantity;

$\delta_{1F}(x)$ is the displacement in the primary system in the direction of removed constraint from the unit force and is the function of argument *x* that is the abscissa of the point of application of the mobile unit force;

$\delta_{F1}(x)$ is a function of the same argument x, but which expresses displacements in the direction of the mobile force from the unit value of the immobile primary unknown $X_1 = 1$, i.e. is a diagram of displacements (is a diagram of deflections of the chord that will be loaded) in the truss with the removed link due to unit value of the force in this link.

Thus, in order to construct a line of influence of a certain effort in a statically indeterminate truss, it is necessary to remove the member perceiving this effort. Then a unit force is applied to the truss with the removed link in the direction of this link. The applied unit force causes the deflections of all nodes of the chord that will be loaded by vertical mobile unit force. The diagram of the displacements of this chord should be constructed (deflection line). The displacement δ_{11} in the direction of the removed link should also be calculated. Usually last displacement is non-zero and positive. Consequently, the ordinates of the deflection line, reduced by δ_{11} of time, are the ordinates of the desired line of influence.

When using computer technology, the influence line for any effort can be built by its direct definition, as a result of the multiple calculation of this effort from the action of a single vertical force equaled to one and applied alternately to each nodes of the chord on which the unit force will move.

If, in one way or another, the influence lines for forces are constructed in all the redundant links of a statically indeterminate truss

$$(Inf.\,Line\ X_k, \quad (k=1,\dots,n)),$$

then the line of the influence for effort in any other rod $(Inf.\,Line\ N^j)$ can be constructed using a simple formula:

$$Inf.\,Line\ N^j = Inf.LineN_0^j + \sum_{k=1}^{n} N_k^j (Inf.LineX_k),$$

where $Inf.Line\ N_0^j$ is the influence line of the force in question in the primary system of the force method with $n = \Lambda$ removed links;

N_k^j is the force in the considered rod in the primary system of the force method from a unit unknown $X_k = 1$.

THEME 13. CALCULATING STATICALLY INDETERMINATE ARCHES, SUSPENSION AND COMBINED SYSTEMS

13.1. Kinds of Statically Indeterminate Arches

The following types of statically indeterminate arches are most commonly used in construction practice: two-hinged arches, single-hinged arches and hingeless arches.

The two-hinged arch (Figure 13.1, a) is characterized by two immovable hinged supports. The single-hinged arch (Figure 13.1, b) contains, as a rule, one hinge in the middle of arch span. The hingeless arch (Figure 13.1, c) represents a continuous curved bar, absolutely rigidly supported at the ends.

From the point of view of static indeterminacy, a two-hinged arch has one “extra” link, a single-hinged arch is twice statically indeterminate, and a hingeless arch is three times statically indeterminate.

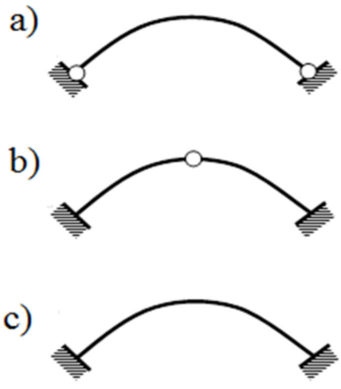


Figure 13.1

As a rule, according to the outline, the arches are symmetrical. Depending on the nature of the load, the axis of the arch can be outlined along a square parabola, along an arc of a circle or other curve, and can be polyline. The cross section of the arch can be either constant or variable along the length of the arch.

All types of arches are thrusting systems, i. e. when a vertical load is applied to an arch, horizontal reactions also arise in its supports.

Therefore, arches require the creation of powerful supporting devices. Arches with ties are used in order not to transmit significant horizontal forces to the underlying structures. Typically, ties of different designs are arranged in two-hinged arches (Figure 13.2). A two-hinged arch with a tie retains the properties of thrust systems, has supports as a simple beam. Therefore, it transfers only vertical forces from a vertical load to the underlying supporting structures and can be located on tall columns or walls without the use of special buttresses

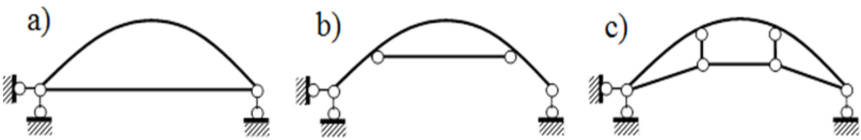


Figure 13.2

The features of calculating once statically indeterminate arches will be considered using the example of calculating a two-hinged arch with a tie.

13.2. Calculation of a Two-Hinged Arch with a Tie

The two-hinged arch with a tie is outwardly non-thrusting. The thrust is perceived by a tie, and should be considered as an internal tensile force in the tie. A two-hinged arch with a tie has only one redundant link and may be easy calculated by the force method.

Let us consider a two-hinged arch with a straight tie located at the level of the supports. The arch has a cross sectional area that may be variable along the span. The arch is loaded with a vertical load (Figure 13.3, a). The primary system of the force method can be obtained by dissecting the tie (more precisely, removing from the tie a link that perceives longitudinal force). The primary unknown of the force method will be the internal tensile force in the tie $N_{tie} = X_1$ (Figure 13.3, b).

The canonical equation of the force method has the form

$$\delta_{11} X_1 + \Delta_{1F} = 0,$$

where δ_{11} is the mutual displacement of the ends of the tie in the cut, caused by the unit effort in the tie $X_1 = 1$, $\Delta_{1F} -$ the mutual displacement of the ends of the cut tie from external loads.

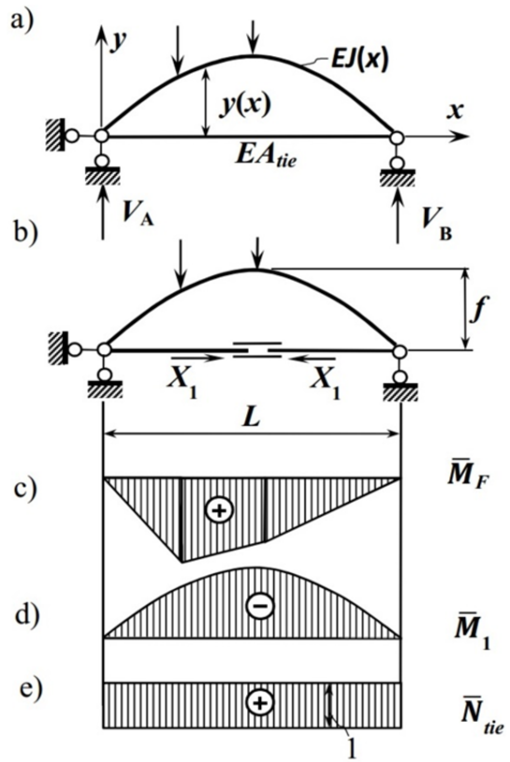


Figure 13.3

When calculating displacements in the primary system, only the bending deformations of the arch and the longitudinal deformations of the tie may be taken into account. Longitudinal and shear deformations in the arch are as a rule neglected. This assumption is valid for arches with a ratio of the arch rise to arch span approximately equal to $f\,/\,L$ = *1/6 ÷ 1/4*.

When the tie is cut, the primary arch system is a curved beam. The load applied to the arch causes bending moments only in the arch as in a curved beam $\bar{M}_F = M_x^0$ (Figure 13.3, c). The load does not cause internal forces in the cut tie.

The unit primary unknown $X_1 = 1$ causes bending moments in the arch $\bar{M}_1 = -y(x)$. The unit diagram of bending moments repeats the outline of the axis of the arch (Figure 13.3, d). The unit primary unknown causes the constant tensile longitudinal force $\bar{N}_{tie} = 1$ in the tie (Figure 13.3, e).

The main feature of calculating arches is that the Mohr integrals for calculating displacements in arches must be taken along the length of the axis of the arch, i.e. they are curvilinear integrals. The free term of the canonical equation is found by the one-term Mohr formula

$$\Delta_{1F} = \int_S \frac{\bar{M}_1 \bar{M}_F ds}{EJ(x)} = -\int_S \frac{y(x)\bar{M}_F ds}{EJ(x)}.$$

The coefficient at the primary unknown (the unit displacement), calculated taking into account the longitudinal deformation of the tie, is found by the two-term formula

$$\delta_{11} = \int_S \frac{\bar{M}_1 \bar{M}_1 ds}{EJ(x)} + \frac{\bar{N}_{tie}^2 L}{EA_{tie}} = \int_S \frac{[y(x)]^2 ds}{EJ(x)} + \frac{L}{EA_{tie}}.$$

To go to the integration over the span, i.e., over the abscissa x, in these curvilinear integrals, it is necessary to introduce the replacement

$$ds = \frac{dx}{\cos\varphi(x)},$$

where $\varphi(x)$ is the angle of inclination to the horizontal of the tangent to the axis of the arch in cross section with the abscissa x. As a result, the following formulas are obtained to calculate the coefficient and the free term of the canonical equation of the force method:

$$\delta_{11} = \int_0^L \frac{[y(x)]^2 dx}{EJ(x)\cos\varphi(x)} + \frac{L}{EA_{tie}};$$

$$\Delta_{1F} = -\int_0^L \frac{y(x)\bar{M}_F dx}{EJ(x)\cos\varphi(x)}.$$

Thus, the calculation of displacements in arches, as well as in other curvilinear bars of variable section, is much more time-consuming than the calculation of displacements in rectilinear bars of constant section. The calculation of definite integrals according to the rule of multiplying diagrams (according to the Vereshchagin rule) is not feasible here, since under the signs of the definite integrals there is a product of several nonlinear functions. The Simpson formula should be considered as one of the options for the approximate (numerical) calculation of the definite integrals.

Taking into account the above assumptions about neglecting longitudinal and shear deformations of the arch, it is permissible to use a simpler method of numerical integration to calculate displacements in the arch. This is the rectangle method. For this purpose, the span of the arch is divided into sufficiently small sections, preferably of the same length. They are numbered in a certain sequence, and in the middle point of each section, the values of all integrand functions are calculated. As a result, the procedure for taking a definite integral is replaced by calculating the final sum of the products of the integrands values in the middle of the sections:

$$\delta_{11} = \sum_{k=1}^{n} \frac{{y_k}^2 \Delta x_k}{EJ_k \cos\varphi_k} + \frac{L}{EA_{tie}};$$

$$\Delta_{1F} = -\sum_{k=1}^{n} \frac{{y_k}^2 (\bar{M}_F)_k \Delta x_k}{EJ_k \cos\varphi_k},$$

where k is the section number, n is the amount of sections. Typically, all calculations are carried out in tables (Table. 13.1).

Table 13.1

№ of section k	x_k	y_k	EJ_k	$\cos\varphi_k$	$(M_F)_k$	Δx_k	$\frac{{y_k}^2 \Delta x_k}{EJ_k \cos\varphi_k}$	$\frac{y_k (M_F)_k \Delta x_k}{EJ_k \cos\varphi_k}$
1	2	3	4	5	6	7	8	9
… … …								
Σ							$\delta_{11}^{(M)}$	$-\Delta_{1F}$

The sum of the elements of the penultimate column gives the part of the unit displacement due to the bending deformations of the arch. The complete unit displacement is found as the sum

$$\delta_{11} = \delta_{11}^{(M)} + \frac{L}{EA_{tie}}.$$

The tie internal force is found from the solution of the canonical equation

$$N_{tie} = X_1 = \frac{-\Delta_{1F}}{\delta_{11}},$$

Internal forces in any section of a two-hinged arch can be found using the same formulas as in a three-hinged arch

$$M_x = M_x^0 - X_1 y_x;$$

$$Q_x = Q_x^0 \cos\varphi_x - X_1 \sin\varphi_x;$$

$$N_x = -(Q_x^0 \sin\varphi_x + X_1 \cos\varphi_x).$$

The index x in these formulas denotes an arbitrary cross section of the arch.

13.3. Influence of the Tie Longitudinal Rigidity on the Tie Internal Force

In statically indeterminate systems, the distribution of internal forces between elements depends on the ratio of their rigidity Therefore, tie deformability (the magnitude $l/(EA_{tie})$) will affect the value of the tie internal force. The formula obtained above for calculating the tie force can be rewritten as follows:

$$N_{tie} = X_1 = \frac{-\Delta_{1F}}{\delta_{11}} = \frac{-\Delta_{1F}}{\delta_{11}^{(M)} + \dfrac{L}{EA_{tie}}}.$$

A graphical representation of the dependence of the tie rigidity on the tie force is given in Figure 13.4.

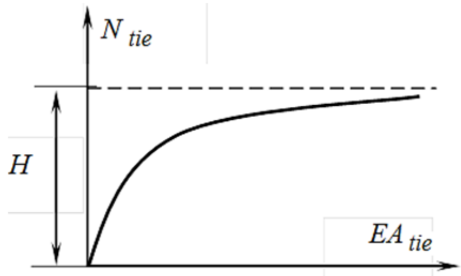


Figure 13.4

If you gradually reduce the tie longitudinal rigidity, the value EA_{tie}, then the tie force will decrease. The weaker the tie, the smaller force it perceives. In the limit, with a tie of zero rigidity, i.e. in the absence of it, the arch with a tie turns into a simple curved beam; tie force is zero.

On the other hand, if the tie longitudinal rigidity is gradually increased, the tie force is also increased, but to a much lesser extent. In the limit, when the tie rigidity tends to infinity the tie force will asymptotically tend to

$$H = \frac{-\Delta_{1F}}{\delta_{11}^{(M)}},$$

where H is a quantity numerically equal to the thrust of a two-hinged arch without a tie on hinged immovable supports (Figure 13.1, a).

That is, in this limiting case, when the tie is absolutely inextensible, the two-hinged arch with the tie turns into a two-hinged arch on hinged immovable supports, as it were, into an ordinary two-hinged arch. Thus, the relatively weak tie does not allow using all the advantages of the arch with the tie as a thrust system.

In contrast, an overly rigid tie is practically useless. The magnitude of the force in a rigid tie cannot exceed the magnitude of the thrust in the ordinary two-hinged arch, calculated by the last formula.

When calculating double-hinged arches without a tie, the thrust is also taken as the primary unknown $(H = X_1)$. The primary system of the force method is obtained by discarding the horizontal support link of one of the supports. The diagram of bending moments due to the primary unknown equal to one and the diagram of load bending moments in the primary system of a two-hinged arch without a tie are the same as for an arch with a tie (Figure 13.3, c, d). The thrust of a two-hinged arch without a tie is calculated by the formula

$$X_1 = H = \frac{\sum_{k=1}^{n} \frac{y_k (M_F)_k \Delta x_k}{EJ_k \cos\varphi_k}}{\sum_{k=1}^{n} \frac{y_k^2 \Delta x_k}{EJ_k \cos\varphi_k}} .$$

13.4. Hingeless Arches. Features of Calculation

A hingeless arch is three times statically indeterminate. To determine the three primary unknowns of the force method, it is necessary to compose and solve three canonical equations. By the appropriate choice of a rational primary system of the method of forces, one can even achieve complete separation of the system of canonical equations into three separate equations, each with one unknown only. This takes place with an arbitrary outline of the axis of the arch and an arbitrary load. Variants of the primary systems shown in Figure 13.5, a, b, c, allow you to reset to zero the secondary coefficients of the canonical equations and bring the equations to the form:

$$\delta_{11} X_1 + \Delta_{1F} = 0 ;$$

$$\delta_{22} X_2 + \Delta_{2F} = 0 ;$$

$$\delta_{33} X_3 + \Delta_{3F} = 0 .$$

Such a result can be obtained at the cost of additional calculations to determine the length of the hard consoles (Figure 13.5, a, b). So, for example, in a symmetric arch (Figure 13.5, b), the primary unknown X_1 is skew-symmetric and is separated from two other direct-symmetric unknowns X_2 and X_3. The direct-symmetric primary unknowns can be separated by selecting the length of absolutely rigid consoles so that the displacements $\delta_{23} = \delta_{32}$ are zero. The point at which the ends of absolutely rigid consoles are located is called the elastic center of the arch. A complete separation of the primary unknowns can also be achieved by placing the end of a single console in the elastic center (Figure 13.5, a).

The same result can be obtained with the primary system in the form of a three-hinged arch (Figure 13.5, c), grouping the primary unknowns X_1 and X_3, and determining the position of the extreme hinges from the conditions so that the secondary coefficients of the canonical equations of the force method vanish.

However, in the age of electronic calculators and computers, solving systems of linear algebraic equations of the second-third order does not present any difficulties. Therefore, you can abandon the choice of the singular primary systems and additional calculations.

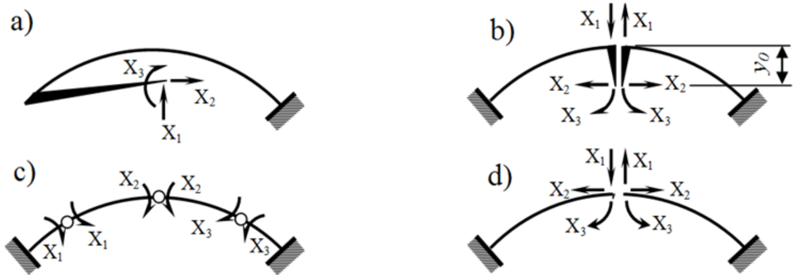


Figure 13.5

So the primary system, obtained by cross-cut of the hingeless arch by the axis of symmetry (Figure 13.5, d), allows you to divide immediately three joint canonical equations into one independent equation relative to the skew-symmetric primary unknown X_1 and into a system of two joint equations relative to two symmetric primary unknowns X_2 and X_3:

$$\delta_{11} X_1 + \Delta_{1F} = 0;$$

$$\delta_{22} X_2 + \delta_{23} X_3 + \Delta_{2F} = 0;$$

$$\delta_{32} X_2 + \delta_{33} X_3 + \Delta_{3F} = 0.$$

The primary system in the form of a curved beam leads to the same results (Figure 13.6, b). Let us consider this option in more detail, since the techniques for constructing a series of diagrams of internal forces in such a primary system have already been reflected in the calculation of two-hinged and three-hinged arches.

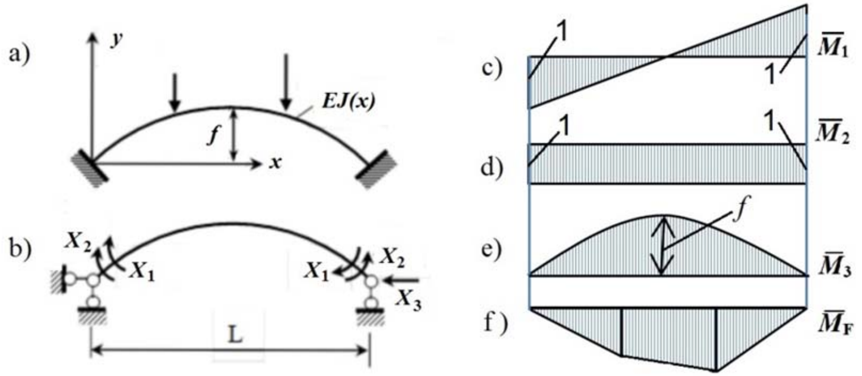


Figure 13.6

We group the unknown support moments, decomposing them into a skew-symmetric group unknown X_1 and a symmetric group unknown X_2. The primary unknown $X_1 = 1$ will cause a linear skew-symmetric diagram of bending moments in the curved beam (Figure 13.6, c). The ordinates of this unit diagram can be calculated in the usual coordinate system with the origin on the left support according to the equation

$$\bar{M}_1(x) = 1 - 2x / L.$$

The unknown $X_2 = 1$ will cause constant positive bending moments in the curved beam (Figure 13.6, d). In the primary system, the unknown $X_3 = 1$ will cause a symmetric diagram of bending moments with negative ordinates, which coincides with the outline of the axis of the arch (Figure 13.6, e). The load diagram of bending moments in the primary system coincides with the beam diagram of bending moments (Figure 13.6, f).

The curvilinear integrals along the length of the arch, determining the coefficients and free terms of the canonical equations, are calculated according to the rule of rectangles, dividing the arch span into n sections. In view of the notation introduced above, we accordingly obtain:

$$\delta_{11} = \sum_{k=1}^{n} \frac{(\overline{M}_1)_k^2 \Delta x_k}{EJ_k \cos\varphi_k}; \qquad \delta_{22} = \sum_{k=1}^{n} \frac{\Delta x_k}{EJ_k \cos\varphi_k};$$

$$\delta_{23} = -\sum_{k=1}^{n} \frac{y_k \Delta x_k}{EJ_k \cos\varphi_k}; \qquad \delta_{33} = \sum_{k=1}^{n} \frac{y_k^2 \Delta x_k}{EJ_k \cos\varphi_k};$$

$$\Delta_{1F} = \sum_{k=1}^{n} \frac{(\overline{M}_1)_k (M_F)_k \Delta x_k}{EJ_k \cos\varphi_k}; \qquad \Delta_{2F} = \sum_{k=1}^{n} \frac{(M_F)_k \Delta x_k}{EJ_k \cos\varphi_k};$$

$$\Delta_{3F} = -\sum_{k=1}^{n} \frac{y_k (M_F)_k \Delta x_k}{EJ_k \cos\varphi_k}.$$

In the above formulas, the index k denotes the section number when calculating the Mohr integrals according to the rectangle rule. The values of the integrands are usually calculated in the middle of the sections.

After determining the primary unknowns from the solution of the canonical equations, one can determine the internal forces in any cross section of the hingeless arch just like in three-hinged and two-hinged arches. The primary unknown X_1 causes vertical support reactions in the primary system

$$V_{A1} = -2X_1/L; \qquad V_{B1} = 2X_1/L.$$

In the primary system the primary unknown X_2 does not cause support reactions. The primary unknown X_3 causes only the horizontal reaction in the primary system

$$H_{A3} = X_3.$$

Therefore, to calculate the internal forces in an arbitrary cross section x of a hingeless arch, we will have the following formulas:

$$M_x = M_x^0 + X_1(1-2x/L) + X_2 - y_x X_3\,;$$

$$Q_x = (Q_x^0 - 2X_1/L)\cos\varphi_x - X_3\sin\varphi_x\,;$$

$$N_x = -(Q_x^0 - 2X_1/L)\sin\varphi_x - X_3\cos\varphi_x\,.$$

13.5. Calculating Combined and Suspension Statically Indeterminate Systems

Remember that design schemes of structures are called combined systems in which some of the bars work on bending, and the rest only on compression-tension. Bars that work on bending usually have a more powerful cross-section and are called rigid members. Rods that accept only compressive or, especially, tensile forces are lighter. They are called flexible elements. Some statically determinate combined systems were considered above in the sixth topic. The types of statically indeterminate combined systems are practically immense.

All types of arches with ties can be attributed to combined systems. An arch itself is a rigid element. Tie elements are flexible members.

Examples of some other statically indeterminate combined systems are shown in Figure 13.8.

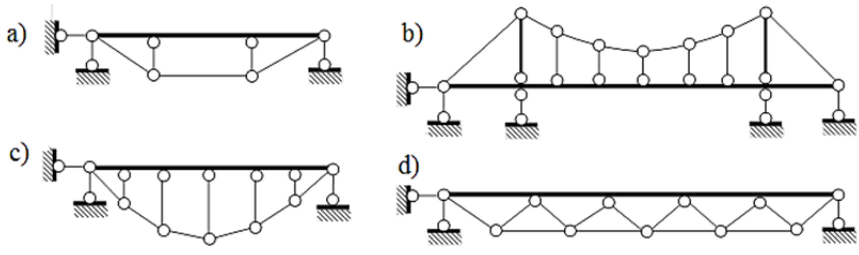


Figure 13.8

Suspension systems are those systems whose main load supporting elements work in tension. Suspension systems include hanging arches

(Figure 13.9, a), various cable-stayed and pure cable systems of bridges, roofs and others constructions (Figure 13.9, b, c, d). Many combined systems can also be attributed to suspension systems (Figure 13.8, b, c).

The calculation of a hanging (stretched) two-hinged arch (Figure 13.9, a) differs from the calculation of an ordinary (compressed) two-hinged arch only in the fact that the thrust of the hanging arch is directed outward from the span. If the hanging arch is made of flexible elements (cables, ropes), then it turns into a flexible thread (Figure 13.9, c). The calculation of flexible threads, as well as other hanging, cable-stayed and combined systems of large spans, is carried out in a nonlinear formulation according to the deformed design scheme. In this section, we consider the features of calculation of some combined systems (including suspension) in the classical linear formulation according to an undeformed design scheme.

The beams with two-post hinged chain (Figure 13.8, a) and with multi-post hinged chain (Figure 13.8, c) are examples of combined systems with one redundant link. A suspension bridge in the form of a chain with a continuous stiffening beam (Figure 13.8, b) is three times statically indeterminate. A truss with a continuous upper chord (Figure 13.8, d) contains four redundant links.

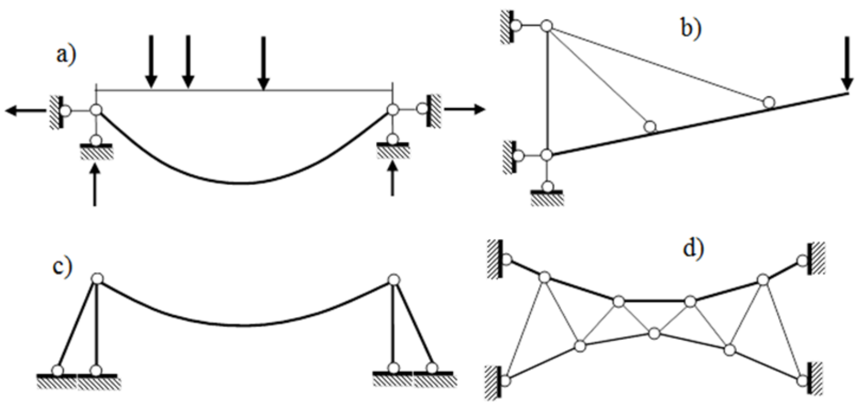


Figure 13.9

The cable-stayed combined system (Figure 13.9, b) with a continuous beam and two cables is statically indeterminate once, provided the cables do not turn off from work. The cable truss (Figure 13.9, d) is statically indeterminate three times. Such trusses must be pre-tensioned so that all their elements are not switched off. Under this condition, the calculation of cable trusses does not differ from the calculation of the usual statically indeterminate trusses considered above.

Calculation of combined systems of small and medium spans with a small number of primary unknowns can be performed by the method of forces according to an undeformed design scheme. It is recommended to choose the primary system of the force method for combined systems so that internal forces from a given load arise only in its rigid elements. This is possible if the entire load is applied to the rigid elements. Figure 13.10 shows options of the primary systems of the force method for two combined systems. The given load and the unit values of primary unknowns cause the bending moments only in their rigid elements. The primary unknowns also cause longitudinal efforts in flexible elements. But there will be no internal forces in the flexible elements of such primary systems from external loads.

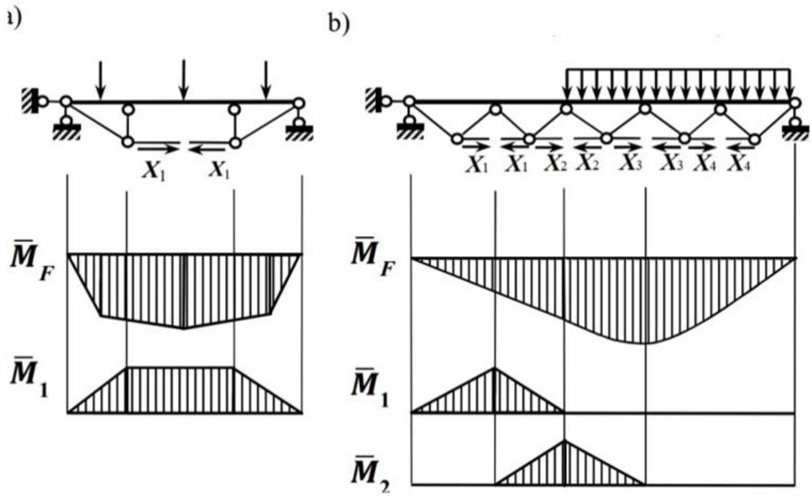


Figure 13.10

The coefficients at the unknowns and the free terms of the canonical equations of the force method in combined systems are calculated by the two-term Maxwell-Mohr formulas:

$$\delta_{ik} = \sum \int \frac{M_i M_k ds}{EJ} + \sum \frac{N_i N_k L}{EA};$$

$$\Delta_{iF} = \sum \int \frac{M_i M_F ds}{EJ} + \sum \frac{N_i N_F L}{EA}.$$

Practically due to the large difference in the cross-sectional areas of flexible and rigid elements, the summation in the second term of the above formulas applies only to flexible elements: corrections to displacements due to longitudinal deformations of rigid elements are obtained insignificant.

THEME 14. GENERAL EQUATIONS OF STRUCTURAL MECHANICS FOR BARS SYSTEMS

14.1. Concept of the Discrete Physical Model

The design schemes of bars systems, used in the classical methods of their analysis, have a pronounced continuity property. They are represented in the form of interconnected one-dimensional elements (bars), while the nodes are interpreted as the points at which the bars are joined or on which constreints (links) are superimposed. As a result of analysing the system for given exposures, dependencies are established. They describe the nature of the change in the internal forces and displacements along the axis of each bar. Information in this form about the stress-deformation state of the system is redundant for practical tasks. In the course of calculations, it is enough to find the internal forces or displacements only in a number of characteristic cross sections and then the forces or displacements can be found in any intermediate section of the bar, if necessary.

Design sections are usually assigned at the junction of the bars to the nodes; they separate the bars from the nodes. As a result, the design scheme of the investigated structure seems to be composed of bars and nodes. This scheme is called the discrete physical model of the structure. For example, the discrete model of the design schemes of a frame (Figure 14.1, a) is shown in Figure 14.1, b.

Each rigid node of a discrete model of a plane structure has three degrees of freedom (linear displacements along the coordinate axes and rotation angle), each hinged one has two degrees of freedom (linear displacements along the coordinate axes). The position of the nodes of the system determines the position of its bars. Therefore, the degree of freedom of a certain system is defined by the number of degrees of freedom of all its nodes.

In this chapter the degree of freedom of the system will be denoted through m . It is precisely the number of independent equations of equilibrium that can be compiled for all nodes of the system.

For plane trusses m is equal to twice the number of nodes minus the number of support rods, and the number of unknown internal forces n is equal to the number of truss rods. Using the notation introduced, the degree of static indeterminacy of the system k can be calculated by the expression

$$k = n - m\,. \tag{14.1}$$

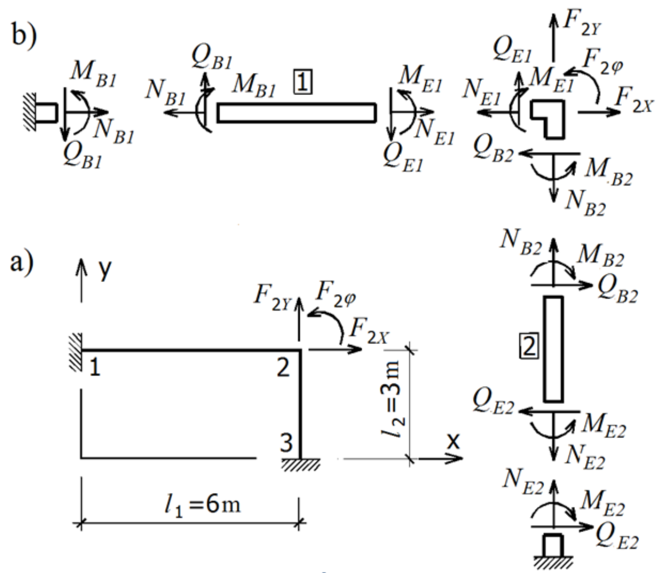


Figure 14.1

In chapter 8, the degree of static indeterminacy of the system was denoted by Λ (the number of redundant links), in chapter 9, the degree of kinematic indeterminacy (it is also the degree of freedom) of the system was denoted by n.

In this chapter, the degree of static indeterminacy is denoted by k, and the degree of kinematic indeterminacy is denoted by m. The number of unknown efforts is indicated through n.

The degree of freedom of a structural node also determines the dimension of the displacement vector of this node. The total number of components of the displacement vector of all nodes corresponds to the degree of kinematic indeterminacy of the system. Therefore, the relation (14.1) can be considered as the relationship between the degree of static indeterminacy k and the degree of kinematic indeterminacy m of the system.

Since any point of any bar can be declared a new node where two bars are joined. Therefore, for the same system, for example, a frame, several variants of its discrete model can be adopted. This means that the degree of freedom of a discrete model, in the general case, is a variable characteristic. Even in this case, relation (14.1) allows one to correctly find the degree of static indeterminacy of the system, since the number of unknown forces in each additional section coincides with the number of independent equilibrium equations that can be compiled for each additional node.

14.2. Loads and Displacements

To simplify the computational procedures, later in this section, the calculation of systems under only nodal forces effects will be considered.

Techniques for replacing distributed load with concentrated forces are well known. The essence of the conversion is as follows.

Initially, each element located between two adjacent nodes is considered as a bar with end (support) links corresponding to the type of a node (rigid or articulated). Calculating it as a single beam under local load, we can determine its support reactions and construct the diagrams of its efforts. For this, table 9.1 can be used.

Subsequently, having loaded the nodes of the design scheme of the system by forces equal in value and opposite in direction to the reactions of the single beams, we will calculate the system on action of these nodal forces.

The final diagrams of the internal forces are obtained by summing the corresponding diagrams from the calculation of the system as a whole and of the individual elements.

Figure 14.2 shows (symbolically) the transition from a system with distributed loads to a system with concentrated forces.

External forces acting on a rigid node i of a plane system are defined by a load vector in the form:

$$\vec{F}_i = \left[\vec{F}_i^x, \vec{F}_i^y, m_i \right]^T .$$

Where $F_i^x, \vec{F}_i^y$ are components of external load along the axes x and y; m_i is a concentrated moment in i-th node.

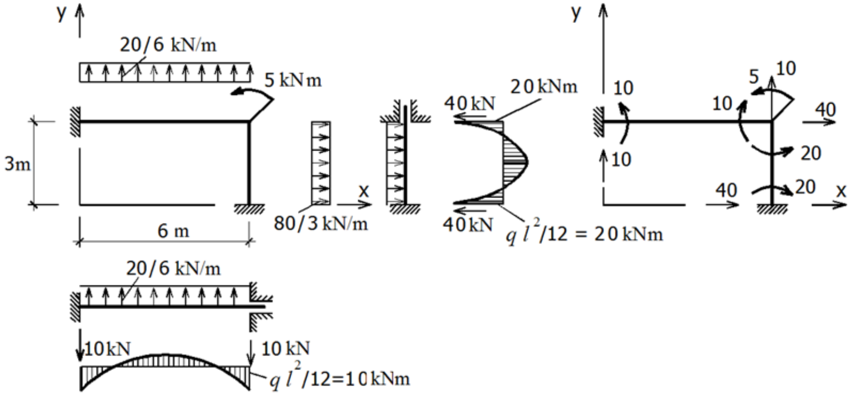


Figure 14.2

The rule of signs for the load: external forces are considered positive if their directions coincide with the directions of the corresponding coordinate axes; positive moments are directed counterclockwise.

The full load vector acting on the system is formed by sequential docking of the corresponding vectors for each node of the system:

$$\vec{F} = \left[\vec{F}_1^T, \vec{F}_2^T, \vec{F}_3^T, \ldots, \vec{F}_{p-1}^T, \vec{F}_p^T \right]^T.$$

The number of system nodes is denoted by p.

Under the load, the system takes a new (deformed) position. Frame nodes move.

The displacements of the rigid node "i" are characterized by a vector

$$\vec{z}_i = \left[\vec{z}_i^{\,x}, \vec{z}_i^{\,y}, \varphi_i \right]^T.$$

The displacements of the articulated node "j" are characterized by a vector

$$\vec{z}_j = \left[\vec{z}_j^{\,x}, \vec{z}_j^{\,y} \right]^T.$$

The full vector of displacements of the system nodes is represented as:

$$\vec{z} = \left[\vec{z}_1^{\,T}, \vec{z}_2^{\,T}, \vec{z}_3^{\,T}, ..., \vec{z}_p^{\,T} \right]^T.$$

The vector of generalized displacements must correspond to the vector of generalized load. Dimensions of vectors $\vec{F}$ and $\vec{z}$ coincide. The scalar product of these vectors determines the work of external forces. Such vectors are called dual.

14.3. Internal Forces (Efforts) and Deformations

Generally, a bending moment M, a transverse force Q and a longitudinal force N arise in the cross section of the bar. Together they form a vector of efforts (internal forces) in the section

$$\vec{S} = [M, Q, N]^T.$$

The components of this vector must be determined.

In special cases, this vector may contain two components, for example:

$$\vec{S} = [M, Q]^T \quad \text{or} \quad \vec{S} = [M, N]^T.$$

In the first case, the longitudinal force is not included in the number of unknowns, and in the second case, the transverse force is.

It is possible that only the bending moment can be an unknown factor in the cross section. Then:

$$\vec{S} = [M].$$

Under the action of the nodal load on the system, the stress state of i^{th} bar can be characterized by the vector:

$$\vec{S}_i = [N_i, M_{Bi}, M_{Ei}, Q_i]^T,$$

where N_i is longitudinal force in the bar;

M_{Bi} – bending moment at the beginning of the bar;

M_{Ei} – bending moment at the end of the bar;

Q_i – transverse force in the bar.

Since the load does not act directly on the rod, the transverse force along its length does not change and, as is known (8.17), is calculated by the formula:

$$Q = \frac{M_E - M_B}{l}.$$

Therefore, having saved the first three components in the vector $\vec{S}_i$, we rewrite it in the form:

$$\vec{S}_i = \left[N_i, M_{Bi}, M_{Ei}\right]^T.$$

Each type of effort corresponds to a certain deformation. The longitudinal force causes elongation or shortening of the element, bending moments – rotations of the cross sections, transverse forces – mutual shear of the cross sections.

The deformation vector $\vec{\Delta}_i$ corresponding to the effort vector $\vec{S}_i$ will have the form:

$$\vec{\Delta}_i = \left[\Delta l_i, \Delta\varphi_{Bi}, \Delta\varphi_{Ei}\right]^T,$$

where Δl_i is linear deformation of an element;

$\Delta\varphi_{Bi}, \Delta\varphi_{Ei}$ are the angles of rotation of the cross sections at beginning and at end of the bar relative to the straight line which connectes the nodes in the deformed state.

The vector of efforts $\vec{S}$ and deformation vectors $\vec{\Delta}$ for the entire system, as well as the vectors $\vec{F}$ and $\vec{z}$, are formed by sequential joining of the efforts and deformation vectors for individual bars.

The vectors $\vec{S}$ and $\vec{\Delta}$ are dual; their scalar product gives the work of internal forces.

For a spatial system, the force vector in the cross section is, as a rule, six-dimensional.

14.4. Equilibrium Equations

Let us consider an arbitrary, for example, statically indeterminate frame (Figure 14.1, a), which is in equilibrium under the action of a given load. The corresponding discrete model in the form of a set of nodes and bars is shown in Figure 14.1, b.

We compose the equilibrium equations for the 2[d] node of the frame:

$$\begin{aligned}
&\sum x = 0, && -N_{E1} - Q_{B2} + F_{2x} && = 0,\\
&\sum y = 0, && +Q_{E1} - N_{B2} + F_{2y} && = 0,\\
&\sum M_2 = 0, && -M_{E1} + M_{B2} + F_{2\varphi} && = 0.
\end{aligned}$$

The written down three equations contain six unknown efforts.

Consider the equilibrium of the frame bars. Each of them is under the action of the end forces shown in the same figure.

Composing the three equilibrium equations for the first bar, we obtain:

$$\begin{aligned}
&\sum x = 0, && N_{B1} = N_{E1} = N_1,\\
&\sum y = 0, && Q_{B1} = Q_{E1} = Q_1,\\
&\sum M_B = 0, && Q_{E1} l_1 + M_{B1} - M_{E1} = 0, \qquad Q_1 = \frac{M_{E1} - M_{B1}}{l_1}.
\end{aligned}$$

Similar relations can be obtained for the second bar, that is:

$$N_{B2} = N_{E2} = N_2, \quad Q_{B2} = Q_{E2} = Q_2, \quad Q_2 = \frac{M_{E2} - M_{B2}}{l_2}.$$

Substituting the expressions of efforts from the equations of equilibrium of the bars into the equations of equilibrium of the nodes, after simple transformations we get:

$$\begin{aligned}
&-N_1 + \frac{1}{l_2} M_{B2} - \frac{1}{l_2} M_{E2} + F_{2x} = 0,\\
&-\frac{1}{l_1} M_{B1} + \frac{1}{l_1} M_{E1} - N_2 + F_{2y} = 0,\\
&-M_{E1} + \quad M_{B2} + \quad F_{2\varphi} = \quad 0.
\end{aligned}$$

The matrix form of this system of equations will be as follows:

$$\begin{bmatrix} -1 & 0 & 0 & \vdots & 0 & \frac{1}{l_2} & -\frac{1}{l_2} \\ 0 & -\frac{1}{l_1} & \frac{1}{l_1} & \vdots & -1 & 0 & 0 \\ 0 & 0 & -1 & \vdots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ M_{B1} \\ M_{E1} \\ N_2 \\ M_{B2} \\ M_{E2} \end{bmatrix} + \begin{bmatrix} F_{2x} \\ F_{2y} \\ F_{2\varphi} \end{bmatrix} = 0$$

or abbreviated:

$$A^{*}\vec{S} + \vec{F} = 0. \tag{14.2}$$

In equations (14.2), the signs of the components of the vector $\vec{F}$ correspond to the accepted positive directions of the nodal loads. To numerically solve these equations, together with others, we rewrite them in the form:

$$A\vec{S} = \vec{F}, \tag{14.3}$$

where $A = -A^{*}$ is equilibrium matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 & -\frac{1}{l_2} & \frac{1}{l_2} \\ 0 & \frac{1}{l_1} & -\frac{1}{l_1} & \vdots & 1 & 0 & 0 \\ 0 & 0 & 1 & \vdots & 0 & -1 & 0 \end{bmatrix};$$

$$\vec{S} = [N_1, M_{B1}, M_{E1}, N_2, M_{B2}, M_{E2}]^{\mathrm{T}}$$

is the vector of effort;

$$\vec{F} = [F_{2x}, F_{2y}, F_{2\varphi}]^{\mathrm{T}}$$

is the vector of loads.

To easily navigate in the structure of matrix A, we remember that in its first line there are the coefficients at unknown efforts in the end sections of the bars from the equilibrium equation $\Sigma X = 0$. In the second - from the equation $\Sigma Y = 0$, and in the third from the equation $\Sigma M_2 = 0$.

Moreover, in the first three columns of the matrix, the coefficients are recorded for the efforts N, M_B, M_E in the first bar, i.e. in the bar 1–2, and in the next three columns for the corresponding efforts in the second bar (in the bar 2–3).

The shown method of forming the equilibrium matrix is called the matrix forming method “by nodes”. For frames with a large number of nodes, it is laborious and therefore is not often used in practice. Another, more effective method, which allows to organize the formation of a matrix “by bars”, will be described in section 14.14.

14.5. Geometric Equations

Let us imagine the process of “transition" of a frame (Figure 14.3) to a deformed state as a result of the successive influence of first longitudinal deformations of its elements, and then bending deformations. The first stage of deformation is equivalent to loading of the corresponding hinged system by nodal forces, which cause the same values of the internal longitudinal forces. Then, with the positions of the nodes at points 1, $2'$ and 3, bending moments are applied which transfare the bars into a curved state. According to the assumption of small displacements, the second stage of deformation does not change the position of the frame nodes. Therefore, the deformation of each bar fixed at the ends can be characterized by three components:

Δl_i is an absolute elongation (shortening) of the i^{th} bar,

$\Delta\varphi_{Bi}, \Delta\varphi_{Ei}$ are angles of rotation of the end sections.

Consequently, the strain vectors of the 1st and 2nd bars of the frame (Figure 14.3) have the form:

$$\vec{\Delta}_1 = \left[\Delta l_1,\ \Delta\varphi_{B1},\ \Delta\varphi_{E1}\right]^{\mathrm{T}},$$

$$\vec{\Delta}_2 = \left[\Delta l_2, \Delta\varphi_{B2}, \Delta\varphi_{E2}\right]^{\mathrm{T}}.$$

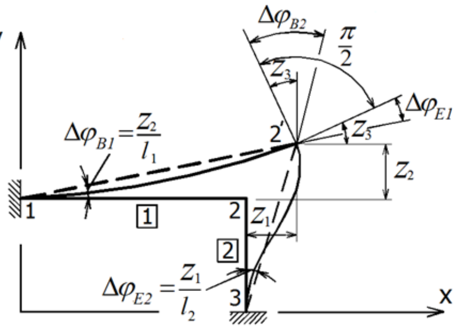


Figure 14.3

Moreover, as follows from Figure 14.3, in which all components of the displacement vector of the 2nd node are shown as positive, the following relations are true:

$$\Delta\varphi_{B1} = \frac{z_2}{l_1}, \qquad \Delta\varphi_{E1} = z_3 - \frac{z_2}{l_1},$$

$$\Delta\varphi_{B2} = -z_3 - \frac{z_1}{l_2}, \qquad \Delta\varphi_{E2} = \frac{z_1}{l_2}.$$

Note. In the expression for $\Delta\varphi_{B2}$ the angle is taken as negative, because the direction of the rotation angle $\Delta\varphi_{B2}$ does not coincide with the direction of the positive moment at the beginning of the 2nd bar.

Due to the smallness of the deformations, we can assume that $\Delta l_1 = z_1$ and $\Delta l_2 = z_2$.

The written relations allow us to establish the relationship between the strain vector $\vec{\Delta}$ and the displacement vector $\vec{z}$ in matrix form:

$$\begin{bmatrix} \Delta l_1 \\ \Delta\varphi_{B1} \\ \Delta\varphi_{E1} \\ \Delta l_2 \\ \Delta\varphi_{B2} \\ \Delta\varphi_{E2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/l_1 & 0 \\ 0 & -1/l_1 & 1 \\ 0 & 1 & 0 \\ -1/l_2 & 0 & -1 \\ 1/l_2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}.$$

Comparing with (14.3), we see that the transformation matrix is transposed with respect to the equilibrium matrix. Hence we can write:

$$\vec{\Delta} = A^T \vec{z}, \tag{14.4}$$

These equations are called geometric equations. They are the equations of continuity of deformations of the bars system.

The matrix A^T is called the deformation matrix. With its help, deformations of system elements are calculated through displacements of nodes.

So that the reader does not have an opinion that the deformation matrix for the considered example turned out to coincide accidentally with the transposed equilibrium matrix, we study the question of the relationship of these matrices in more detail.

14.6. Duality Principle

The equilibrium equations were compiled for the undeformed state of the system, that is, under the assumption of small deformations of its elements, causing small displacements of nodes.

Due to this assumption, the equilibrium equations and geometric equations turned out to be linear. Systems to which this assumption applies are called geometrically linear.

An important property of equations is that the matrices of equilibrium equations and geometric equations are mutually transposed. This relationship can be shown in general terms. Let, for example, between vectors $\vec{z}$ and $\vec{\Delta}$ there is dependence in the form:

$$A_1 \vec{z} = \vec{\Delta}.$$

In accordance with the virtual displacement principle for a system in equilibrium, the sum of the virtual works of external and internal forces is zero. Actual displacements can be considered as a special case of virtual. In this case:

$$F^T \vec{z} - S^T \vec{\Delta} = 0. \tag{14.5}$$

Substituting

$$F^T = S^T \ A^T \ \text{and} \ \vec{\Delta} = A_1 \ \vec{z},$$

into the equations (14.5), we will have

$$S^T \ A^T \ \vec{z} - S^T \ A_1 \ \vec{z} = 0,$$

which implies the equality

$$A_1 = A^T.$$

The obtained dependence is common for linear systems and expresses a static-geometric analogy of the calculated relations.

In the case of large displacements, the problem of determining the stress-strain state becomes nonlinear. Systems in which large displacements and small deformations take place, together with the corresponding problems are called geometrically nonlinear. An example of geometrically nonlinear systems can be some cable-stayed systems. For these systems, equilibrium equations are compiled for their deformed state taking into account nodal displacements. The matrix A of equilibrium equations will depend on the displacements z, the matrix A_1 of geometric equations will also be dependent on z, but

$$\left[A(z)\right]^T \neq A_1(z).$$

The static-geometric analogy for geometrically non-linear problems appears in a more complex form. Its consideration is beyond the scope of this tutorial.

In the following presentation of the theory of calculating bars systems, geometrically linear systems are considered.

14.7. Physical Equations

The relationship between the strain vector and the force vector for an individual bar is established linear:

$$\vec{\Delta}_i = D_i \ \vec{S}_i \, .$$

Recall that:

$$\vec{\Delta}_i = \left[\Delta l_i, \Delta\varphi_{Bi}, \Delta\varphi_{Ei}\right]^T, \qquad \vec{S}_i = \left[N_i, M_{Bi}, M_{Ei}\right]^T.$$

To determine the component $\Delta\varphi_{Bi}$, one should consider loading the bar with the end moments M_{Bi} and M_{Ei} (load state) and loading with a unit moment at the beginning of the bar (Figure 14.4). "Multiplying" the diagrams, we get:

$$\Delta\varphi_{Bi} = \frac{2l}{6(EJ)_i} M_{Bi} + \frac{l}{6(EJ)_i} M_{Ei}.$$

Similar reasoning will allow us to write the expression:

$$\Delta\varphi_{Ei} = \frac{l}{6(EJ)_i} M_{Bi} + \frac{2l}{6(EJ)_i} M_{Ei}.$$

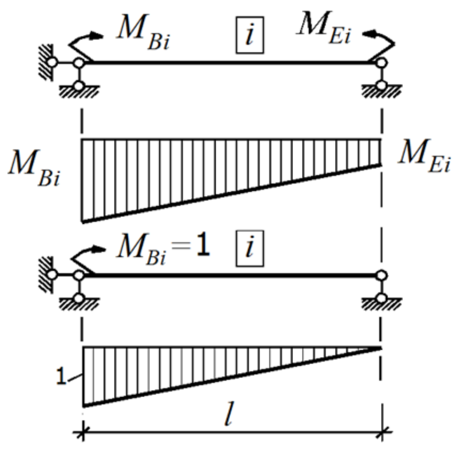


Figure 14.4

Given that the compliance of the bar from the longitudinal force is equal to $l_i / (EA)_i$, the matrix of internal compliance of the bar, taking into account tensile-compression and bending deformations, will be as follows:

$$D_i = \begin{bmatrix} \dfrac{l_i}{(EA)_i} & & \\ & \dfrac{2l_i}{6(EJ)_i} & \dfrac{l_i}{6(EJ)_i} \\ & \dfrac{l_i}{6(EJ)_i} & \dfrac{2\,l_i}{6(EJ)_i} \end{bmatrix}.$$

The frame (Figure 14.3) consists of two bars, so the matrix of internal compliance of the system is quasi-diagonal:

$$D = \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix},$$

and the physical equations are written in the form:

$$\vec{\Delta} = D\,\vec{S}.$$

14.8. Features of the Calculation of the Systems for Temperature Changes, Settlements of Supports and Inaccuracy in the Manufacture of Bars

These factors are taken into account by appropriate adjustment of physical equations. The deformations vector $\vec{\Delta}$ caused by the efforts from the load $\vec{F}$ should be summed with the new deformations vector $\vec{\Delta}'$ from other exposures.

As the temperature changes with respect to a certain initial state, the frame bars become deformed (Figure 14.5). Denote by t_1 the temperature change along the upper face of the bar, through t_2 – along the bottom.

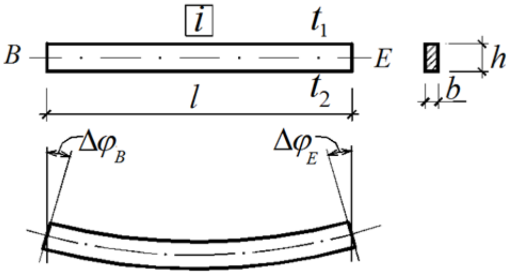


Figure 14.5

Let

$$t_2 > t_1.$$

A change in temperature along the axis of the rod

$$t = \frac{t_1 + t_2}{2}$$

causes its extension

$$\Delta l = \alpha\, t\, l.$$

The temperature difference

$$t' = t_2 - t_1$$

causes the rotation of the end sections by angles determined by the formula (7.12):

$$\Delta\varphi_B = \Delta\varphi_E = \frac{\alpha\, t'}{h}\frac{l}{2}.$$

The directions of rotation of the end sections of the bar in Figure 14.5 are shown as positive. In this case, the strain vector for this bar is written as follows:

$$\Delta'_{t_i} = \left[\Delta l_i,\ \Delta\varphi_{Bi},\ \Delta\varphi_{Ei}\right]^{\mathrm{T}}.$$

For the entire system a strain vector is formed by joining vectors for individual bars.

When calculating the system for the inaccuracy of manufacturing its elements, the vector $\vec{\Delta}'_B$ is known by the condition of the problem. The components of the strain vector from inaccurate production of bars are determined by the difference between the real and design values of the dimensions of the bars.

The vector of deformation of the bars from the settlement of the supports can be obtained as follows. We select from the matrix A the rows associated with the equilibrium conditions of the support nodes in the directions of the support links. Displacements can have all support nodes or only a part of them. The number of such lines, equal to the number of support links, is denoted by r. The corresponding equilibrium conditions for the support nodes of the system are written in the form:

$$A^{(r)}\,\vec{S} = 0.$$

We will divide the matrix $A^{(r)}$ into blocks using a vertical partition (Table 14.1) and we will consider it as a complex matrix $A^{(r)} = \left[A_{n-r}^{(r)},\ A_r^{(r)} \right]$.

Table 14.1

	S_1	S_2	…	S_{n-r}	S_{n-r+1}	·	S_n
1	×	×	…	×	1		
⋮	⋮	⋮		⋮		⋮	
r	×	×	…	×			1

The matrix $A_{n-r}^{(r)}$ is of type $r \cdot (n-r)$, and the matrix $A_r^{(r)}$ is of $r \cdot r$.

The forces $S_{n-r+1},..., S_n$ are equal to the reactions in the support links:

$$\vec{S}_r = \vec{R}\,.$$

The equilibrium equations for the support nodes can be written as follows:

$$A_{n-r}^{(r)}\ \vec{S}_{n-r}+A_{r}^{(r)}\ \vec{S}_{r}=0\ .$$

Since $A_r^{(r)}$ is the identity matrix, then:

$$\vec{R}=\vec{S}_r=-A_{n-r}^{(r)}\ \vec{S}_{n-r}\ .$$

For given displacements of the support links, the deformation vector of the rods is determined by the expression:

$$\vec{\Delta}'_c=-A_{n-r}^{(r)^{\mathrm{T}}}\ \vec{z}\ .$$

The order of the vector $\vec{z}$ is r .

To take into account the considered exposures, the physical equations should be written in the form:

$$\vec{\Delta}=D\,\vec{S}+\vec{\Delta}'\ . \qquad (14.6)$$

14.9. Calculating the Bar Systems. General Equations. The Mixed Method

Equations of equilibrium (14.3), geometric (14.4) and physical equations (14.6) together form a common system of equations for calculating a linearly deformable bars system. Imagine them in the following form:

$$\begin{cases} A\ \vec{S}=\vec{F}; \\ A^{\mathrm{T}}\ \vec{z}-\vec{\Delta}=0; \\ \vec{\Delta}-D\ \vec{S}=\vec{\Delta}'. \end{cases} \qquad (14.7)$$

The sought-for quantities in (14.7) are the n – dimensional force vector $\vec{S}$, the m – dimensional displacement vector $\vec{z}$, and the n -dimensional strain vector $\vec{\Delta}$. Total unknowns – $(2n+m)$. The number of equations in the system is also equal to $2n+m$: equilibrium equations – n, geometric equations – m, physical equations – n. Therefore, this common system of linear independent equations has a unique solution. This means that the exposures $\vec{F}$ and $\vec{\Delta}'$ acting on the structure, according to the solution of the system of equations, cause a single picture of the distribution of forces, displacements and deformations in it. Such a system of determining mathematical relationships is called the mathematical model for calculating the bars system.

The order of the system of equations (14.7) can be reduced. For example, if we find the strain vector $\vec{\Delta}$ from the third group of equations and substitute it into the second group of equations, then the system of equations (14.7) is transformed to:

$$\begin{cases} A\,\vec{S} = \vec{F}; \\ A^{\mathrm{T}}\,\vec{z} - D\ \vec{S} = \vec{\Delta}', \end{cases} \tag{14.8}$$

or in matrix form:

$$\begin{bmatrix} A & 0 \\ -D & A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \vec{S} \\ \vec{z} \end{bmatrix} = \begin{bmatrix} \vec{F} \\ \vec{\Delta}' \end{bmatrix}. \tag{14.9}$$

The forces and displacements are unknown in this version of the mathematical model. Therefore, the system of equations of the form (14.8) or (14.9) is called the system of equations of the mixed method.

Solving the equations of the mixed method allows you to find the forces in the bars of the system and the displacements of its nodes.

14.10. Displacement Method

We represent the equations of equilibrium in displacements. If there are no infinitely rigid elements in the bars system, the quasi-diagonal matrix D is a nonsingular matrix; its determinant is nonzero. Therefore, from the second group of equations (14.8) we can find the vector $\vec{S}$:

$$\vec{S} = D^{-1}\left(A^{\mathrm{T}}\,\vec{z} - \vec{\Delta}'\right) = K\left(A^{\mathrm{T}}\,\vec{z} - \vec{\Delta}'\right),$$

where K is the matrix of the internal stiffness of the bars system.

Substituting $\vec{S}$ in the first group of equations (14.8), we obtain a record of equilibrium equations through displacements $\vec{z}$ in the form:

$$AK(A^{\mathrm{T}}\vec{z}-\vec{\Delta}')=\vec{F} \quad \text{or} \quad AK\,A^{T}\,\vec{z}=\vec{F}+AK\vec{\Delta}'.$$

This system of equations is the system of equations of the displacement method. We introduce the notation

$$R=AK\,A^{T}$$

and rewrite it in this form:

$$R\vec{z}=\vec{F}+\ AK\vec{\Delta}'. \tag{14.10}$$

Matrix R is a matrix of external stiffness of an elastic system; it has a size $(m\cdot m)$.

As follows from the scheme of calculating, the matrix is symmetric with respect to the main diagonal. The elements of the matrix are determined taking into account the influence of longitudinal and bending deformations. If the calculation is carried out only on the action of the load, that is, if the vector of forced deformations $\vec{\Delta}'=0$, then the system of equations of the displacement method is written in the form:

$$R\vec{z}=\vec{F}, \tag{14.11}$$

or in expanded form:

$$\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ r_{m1} & r_{m2} & \cdots & r_{mm} \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{bmatrix}.$$

By definition, r_{ik} is the force (reaction) in the i-th additional link due to the displacement $z_k=1$.

E x a m p l e. Calculate the frame shown in Figure 14.6. The ratio of the rigidity of the bars in tension (compression) and bending is taken equal to

$$\frac{EA\,h^2}{EJ}=1,\ (h\ =1\ \text{m}).$$

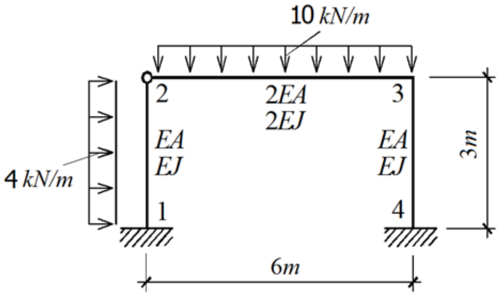


Figure 14.6

First of all, we transform the given load to the nodal one. The end reactions in single-span statically indeterminate beams loaded with a distributed load, and the outline of the bending moment diagrams in them (Figure 14.7), we will find using Table. 9.1. Then the design scheme of the frame with a nodal load can be represented as it is shown in Figure 14.8.

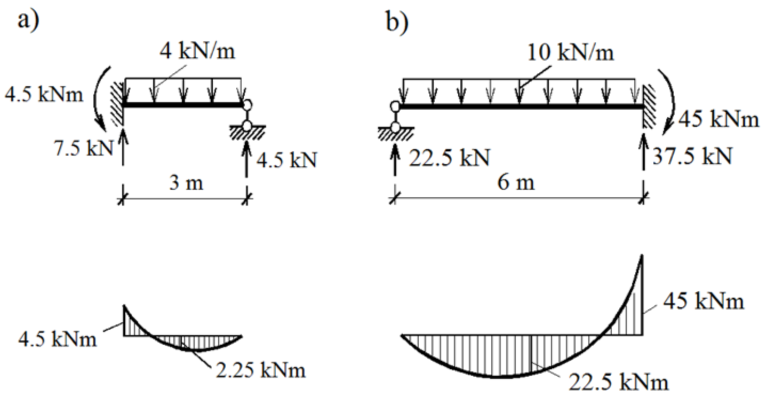


Figure 14.7

The primary system for calculating the frame, taking into account the longitudinal deformations of the bars, is shown in Figure 14.9. The positive directions of the primary unknowns comply with the sign rule specified in section 14.2.

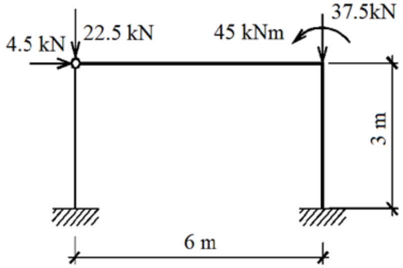


Figure 14.8

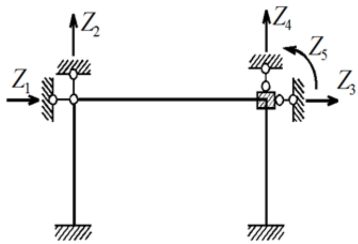


Figure 14.9

Using the equilibrium matrix and the matrix of internal stiffness of the frame, we calculate the matrix of external stiffness:

$$R = AKA^T .$$

$$R = \begin{bmatrix} & -1/3 & -1 & & & & \\ 1 & & & 1/6 & & & \\ & & 1 & & & -1/3 & 1/3 \\ & & & -1/6 & 1 & & \\ & & & 1 & & -1 & \end{bmatrix} \times$$

$$\times \begin{bmatrix} 1/3 & & & & & & \\ & 1 & & & & & \\ & & 1/3 & & & & \\ & & & 1 & & & \\ & & & & 1/3 & & \\ & & & & & 4/3 & -2/3 \\ & & & & & -2/3 & 4/3 \end{bmatrix} \cdot \begin{bmatrix} & 1 & & & \\ -1/3 & & & & \\ -1 & & 1 & & \\ & 1/6 & & -1/6 & 1 \\ & & & 1 & \\ & & -1/3 & & -1 \\ & & 1/3 & & \end{bmatrix} \cdot EJ =$$

$$= \begin{bmatrix} 0.4444 & & -0.3333 & & \\ & 0.3611 & & -0.0278 & 0.1667 \\ -0.3333 & & 0.7778 & & 0.6667 \\ & -0.0278 & & 0.3611 & -0.1667 \\ & 0.1667 & 0.6667 & -0.1667 & 2.3333 \end{bmatrix} \cdot EJ .$$

The load vector $\vec{F}$ in the equation of the form $R\,\vec{z} = \vec{F}$ corresponds to the load shown in Figure 14.8:

$$\vec{F} = [4.5; -22.5; 0; -37.5; 45.0]^T .$$

Having solved the system of equations of the displacement method, we obtain:

$$\vec{z} = [-9.184; -81.619; -25.745; -98.381; 25.444]^T \cdot \frac{1}{EJ} .$$

The force vector calculated by the expression $\vec{S} = K\,A^T\vec{z}$ can be written as:

$$\vec{S} = [-27.21; 3.06; -5.52; 28.24; -32.79; -16.76; -0.20]^T .$$

Figures 14.10, a, b show the diagrams of the frame efforts corresponding to this vector.

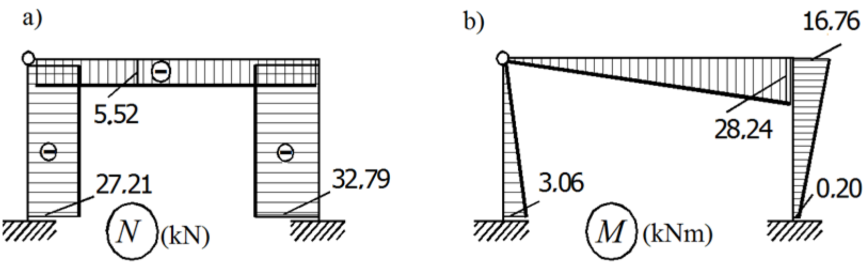


Figure 14.10

By superimposing the diagrams of bending moments in the beams (Figure 14.7) on diagram M (Figure 14.10, b) we obtain the final diagram M in the frame (Figure 14.11).

Naturally for a different initial ratio of rigidities EA/EJ, the ordinates of the diagram M will differ from those found.

To assess the effect of longitudinal deformations on the distribution of displacements and forces in the frame, we will perform its calculation taking into account only bending deformations (Figure 14.12, a). Neglecting longitudinal deformations, we choose the primary system of the displacement method with two unknowns (Figure 14.12, b).

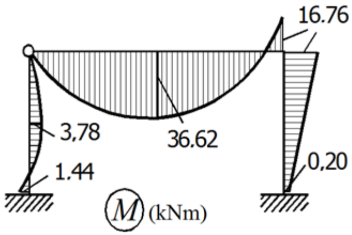


Figure 14.11

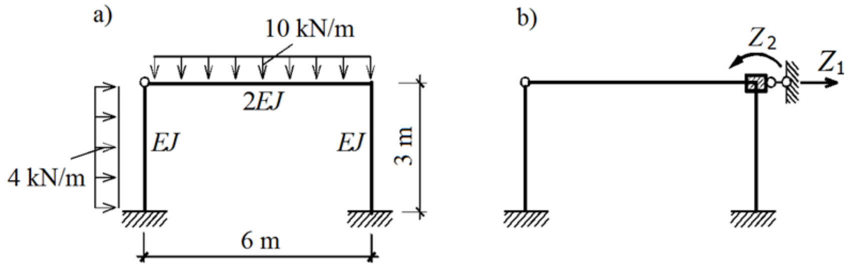


Figure 14.12

Having completed the necessary steps of the calculations, we find the values of the primary unknowns:

$$z_1 = 22.891\frac{1}{EJ}\ \text{m}, \qquad z_2 = 25.826\frac{1}{EJ}\ \text{rad}.$$

The diagram of bending moments is shown in Figure 14.13.

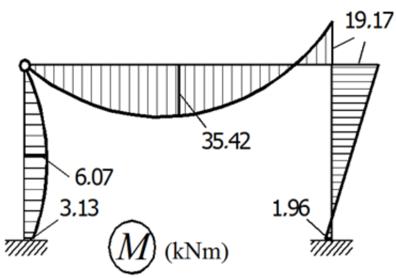


Figure 14.13

14.11. Force Method

For a statically indeterminate system the number of equilibrium equations (m) is less than the number of unknown efforts (n). The equilibrium matrix A has dimensions ($m \cdot n$).We will renumber unknown efforts so that the last numbers (out of the total number n) are assigned to those efforts that are accepted as the primary unknowns of the method of forces. Then we divide the matrix into two submatrices A_0 and A_x :

$$A = \begin{bmatrix} A_0 & A_x \end{bmatrix},$$

where A_0 is the equilibrium matrix of the primary system, $\det A_0 \neq 0$.

The forces in the bars of the adopted primary system are denoted by $\vec{S}_0$.

The matrix A_x contains those columns of the matrix A that correspond to the primary unknowns $\vec{X}$ of the force method.

We write the equations of equilibrium in the following form:

$$A_0 \, \vec{S}_0 + A_x \, \vec{X} = \vec{F} \; .$$

Geometric and physical equations

$$A^{\mathrm{T}} \, \vec{z} = \vec{\Delta} \, ,$$

$$\vec{\Delta} = D \, \vec{S} + \vec{\Delta}' \, ,$$

after converting them to kind

$$A^{\mathrm{T}} \, \vec{z} - D \, \vec{S} = \vec{\Delta}'$$

and taking into account the block recording of the matrix A, can be represented as follows:

$$\begin{bmatrix} A_0^T \\ A_x^T \end{bmatrix} \begin{bmatrix} \vec{z} \end{bmatrix} - \begin{bmatrix} D_0 & 0 \\ 0 & D_x \end{bmatrix} \begin{bmatrix} S_0 \\ X \end{bmatrix} = \begin{bmatrix} \Delta_0' \\ \Delta_x' \end{bmatrix}.$$

The vector-matrix recording of these operations is reduced to two subsystems of equations:

$$A_0^{\mathrm{T}}\, \vec{z} - D_0\, \vec{S}_0 = \vec{\Delta}_0',$$

$$A_x^{\mathrm{T}}\, \vec{z} - D_x\, \vec{X} = \vec{\Delta}_x'.$$

Thus, a system of equations of mixed form can be written in the form of three equations:

$$A_0\, \vec{S}_0 + A_x\, \vec{X} = \vec{F},$$

$$A_0^{\mathrm{T}}\, \vec{z} - D_0\, \vec{S}_0 = \vec{\Delta}_0',$$

$$A_x^{\mathrm{T}}\, \vec{z} - D_x\, \vec{X} = \vec{\Delta}_x'.$$

We exclude the vectors $\vec{S}_0$ and $\vec{z}$ from this system. It follows from the first equation that:

$$\vec{S}_0 = A_0^{-1}\left(-A_x\, \vec{X} + F\right) = L_x\, \vec{X} + \vec{S}_F^0, \tag{14.12}$$

where

$$L_x = -A_0^{-1}\, A_x, \qquad \vec{S}_F^0 = A_0^{-1}\, \vec{F};$$

$A_0^{-1} = L_S$ is influence matrix of efforts in the bars of the primary system, constructed from the action of unit forces oriented along the directions of nodal loads;

$\vec{S}_F^0 = A_0^{-1}\, \vec{F}$ is efforts vector in the bars of the primary system from the load $\vec{F}$;

L_x is influence matrix of efforts in the bars of the primary system, constructed from the action of unit forces oriented in the directions of the primary unknowns;

$L_x\, \vec{X}$ is efforts in the bars of the primary system from $\vec{X}$.

From the second equation we find $\vec{z}$:

$$\vec{z} = \left(A_0^{-1}\right)^{\mathrm{T}} D_0\, \vec{S}_0 + \left(A_0^{-1}\right)^{\mathrm{T}} \vec{\Delta}_0{}' . \tag{14.13}$$

Substituting $\vec{S}_0$ into the last expression and then $\vec{z}$ into the third equation, we obtain the equations of the force method in the following form:

$$\left(L_x^{\mathrm{T}} D_0\, L_x + D_x\right) \vec{X} + L_x^{\mathrm{T}} D_0\, \vec{S}_F^0 + L_x^{\mathrm{T}} \vec{\Delta}_0{}' + \vec{\Delta}_x{}' = 0\,, \tag{14.14}$$

where $D_0\, L_x\, \vec{X}$ is deformation of the bars of the primary system from efforts $\vec{X}$;

$D_0\, \vec{S}_F^0$ is deformation of the bars of the primary system from forces $\vec{F}$.

Having determined $\vec{X}$, one can find the forces in the bars belonging to the primary system by the formula (14.12), and then, by the formula (14.13), the vector of nodal displacements $\vec{z}$.

E x a m p l e. We'll show the calculation of the truss (Figure 14.14) by the force method. The cross-sectional areas of all the rods are assumed to be the same and equal to $A = 0.25\, m^2$. The elastic modulus of the material $E = 210\, GPa$.

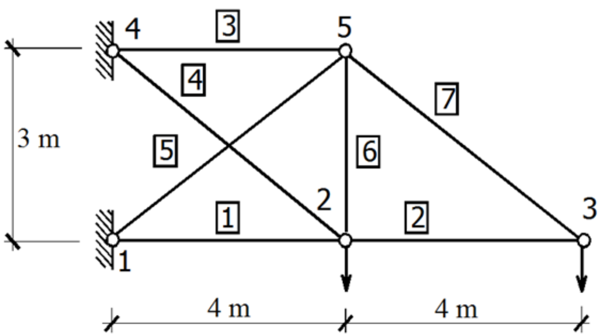


Figure 14.14

The equilibrium matrix of this truss, taking into account the accepted numbering of nodes and rods, has the following form:

A =	1	-1	0	08	0	0	0
	0	0	0	-0.6	0	-1	0
	0	1	0	0	0	0	0.8
	0	0	0	0	0	0	-0.6
	0	0	1	0	0.8	0	-0.8
	0	0	0	0	0.6	1	0.6

The degree of static indeterminacy of the truss is $k = n - m = = 7 - 6 = 1$.

One of the main conditions for choosing the primary system of the method of forces is, as you know, the condition for its geometric invariability. The determinant of the equilibrium matrix of the primary system should not be zero.

We take the force in the 4th rod as the primary unknown. Then the equilibrium matrix of the primary system (Figure 14.15) will have the following form:

$\boldsymbol{A_0}$ =	1	-1	0	0	0	0
	0	0	0	0	-1	0
	0	1	0	0	0	0.8
	0	0	0	0	0	-0.6
	0	0	1	0.8	0	-0.8
	0	0	0	0.6	1	0.6

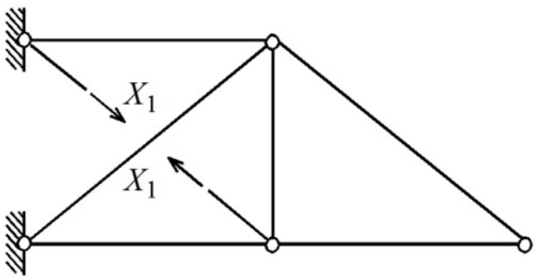


Figure 14.15

The determinant of this matrix is equal to $\det A_0 = \text{-}0.36$.

The matrix A_x is represented by the fourth column of the equilibrium matrix of the given system:

$$A_x = [0.8\,; -0.6\,; 0\,; 0\,; 0\,; 0]^T\,.$$

We find the matrix inverse to the matrix A_0 and the influence matrix L_x for the efforts in the rods of the primary system from $X_1 = 1$:

$$A_0^{-1} = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 0 & 1 & 1.333 & 0 & 0 \\ \hline 0 & 0 & 1 & 1.333 & 0 & 0 \\ \hline 0 & -1.333 & 0 & -2.667 & 1 & -1.333 \\ \hline 0 & 1.667 & 0 & 1.667 & 0 & 1.667 \\ \hline 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1.667 & 0 & 0 \\ \hline \end{array} \qquad L_x = -A_0^{-1}\,A_x = \begin{array}{|c|}\hline -0.8 \\ \hline 0 \\ \hline -0.8 \\ \hline 1 \\ \hline -0.6 \\ \hline 0 \\ \hline \end{array}\,.$$

The efforts $\vec{N}_F^0$ in the rods of the primary system from the load

$$\vec{F} = [0; -10.0; 0; -10.0; 0; 0]^T$$

will have the following values:

$$\vec{N}_F^0 = A_0^{-1}\,\vec{F} = [-13.333;\ -13.333;\ 40;\ -33.333;\ 10;\ 16.667]^T.$$

The matrix of internal compliance of the rods of the primary system is diagonal and is represented in such a record:

$$diag\,D_0 = \left[\frac{l_1}{E\,A_1}; \frac{l_2}{E\,A_2}; \frac{l_3}{E\,A_3}; \frac{l_5}{E\,A_5}; \frac{l_6}{E\,A_6}; \frac{l_7}{E\,A_7}\right] =$$

$$= [0.7619; 0.7619; 0.7619; 0.9524; 0.5714; 0.9524] \cdot 10^{-4}.$$

The compliance of the rod, the force in which is taken as the primary unknown X_1, is equal to $D_x = \dfrac{l_4}{E\,A_4} = 0.9524 \cdot 10^{-4}$.

The matrix of compliance of the primary system in the directions of the primary unknowns in the case of one unknown is represented by one element:

$$D = L_x^T \, D_0 \, L_x + D_x = 30.857 \cdot 10^{-5} \, .$$

The displacement in the direction of the primary unknown, caused by a given load, that is, a free term in the canonical equation of the method of forces, is:

$$\Delta = L_x^T \, D_0 \, \vec{N}_F^0 = -514.286 \cdot 10^{-5} \, .$$

From equation (14.14), which can be written in the form:

$$D \, X_1 + L_x^T \, D_0 \, \vec{N}_F^0 = 0 \, ,$$

we find

$$X_1 = 16.667 \, \text{kN}.$$

Using expression (14.12), we determine the final forces in all the rods of the primary system of the given truss:

$$\vec{N}_0 = L_x \, X_1 + \vec{N}_F^0 =$$
$$= \left[-26.67; -13.33; 26.67; \; -16.67; 0; 16.67 \right]^T \; \text{кN},$$

and according to the expression (14.13) – the displacements of its nodes:

$$\vec{z} = \left(A_0^{-1} \right)^T \, D_0 \, \vec{N}_0 =$$
$$= \left[-0.2032; -0.5355; -0.3048; -1.477; 0.2032 \, ; -0.5355 \right]^T \cdot 10^{-2} \; \text{m}.$$

14.12. Statically Determinate Systems

In a statically determinate system, the number of independent equilibrium equations is equal to the number of unknown efforts; therefore, the equilibrium matrix A is square. In this case, the system of equations (14.7) splits into two independent groups of equations.

From the first of them it follows that if $\det A \neq 0$, then:

$$\vec{S} = A^{-1}\,\vec{F}\,.$$

The condition that the determinant of the matrix A is equal to zero is a sign that the calculated system is partially geometrically variable or instantly variable.

The second group of equations allows you to calculate the displacement vector $\vec{z}$:

$$\vec{z} = \left(A^{-1}\right)^T \left(D\,\vec{S} + \vec{\Delta}'\right).$$

In the absence of external load, we obtain:

$$\vec{S} = 0 \quad \text{and} \quad \vec{z} = \left(A^{-1}\right)^T \vec{\Delta}'\,.$$

These relations confirm the well-known property of statically determinate systems: a change in temperature, displacements of supports or inaccuracy in the manufacture of elements in statically determinate systems do not cause internal forces, but cause only displacements.

14.13. General Equations for a Bar

Consider a frame loaded with a nodal load (Figure 14.16, a), and a fragment of its discrete scheme (Figure 14.16, b). The directions of the nodal forces and the forces of interaction in the sections shown in the figure correspond to the directions of the axes of the general coordinate system.

We establish the relationship of the load in the nodes i, j and the efforts in the sections adjacent to the nodes. This dependence is easier to obtain first in the local coordinate system (for the bar $i-j$ – the system $\xi\eta$), and then, using the rules of linear transformations, in the general system XY.

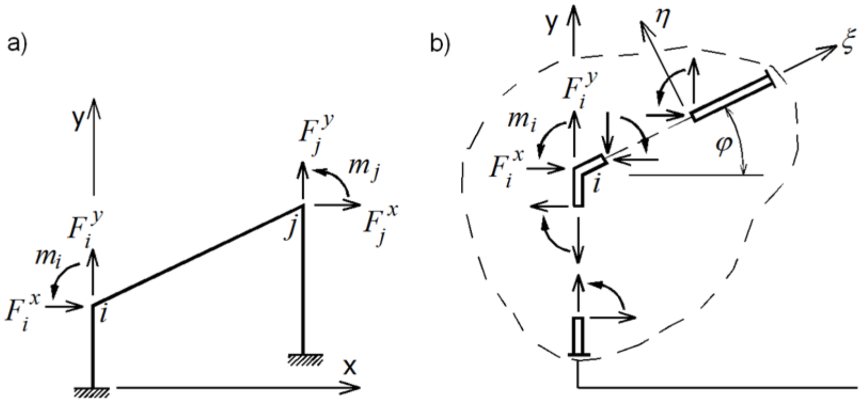


Figure 14.16

The directions of efforts in the end sections of the bar and nodal forces oriented along the axes of the local coordinate system are shown in Figures 14.17, a, b, c.

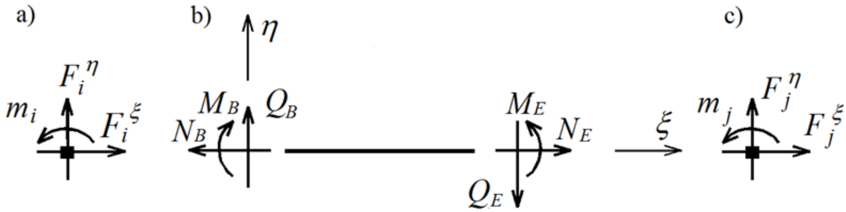


Figure 14.17

In general, the efforts vectors at the beginning of the bar

$$\vec{S}_B = [N_B, Q_B, M_B]^T$$

and at the end of it

$$\vec{S}_E = [N_E, Q_E, M_E]^T$$

contain three components each. In relation to the bar, these forces are external and dependent; they are connected by three equations of equilibrium:

$$\sum \xi = 0, \qquad -N_B + N_E = 0, \qquad N_B = N_E = N;$$
$$\sum \eta = 0, \qquad Q_B - Q_E = 0, \qquad Q_B = Q_E = Q;$$
$$\sum M_B = 0, \qquad M_B - M_E + Q\,l = 0, \qquad Q = \frac{1}{l}\left(M_E - M_B\right).$$

If the stress state of the bar is characterized by a vector $\vec{S} = \left[N, M_B, M_E\right]^T$, then it is necessary to establish the relationship between $\vec{S}_B, \vec{S}_E$ and $\vec{S}$. In the matrix form of recording, this dependence is determined in this way:

$$\begin{bmatrix} \vec{S}_B \\ \vec{S}_E \end{bmatrix} = \begin{bmatrix} N_B \\ Q_B \\ M_B \\ N_E \\ Q_E \\ M_E \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/l & 1/l \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1/l & 1/l \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} N \\ M_B \\ M_E \end{bmatrix}.$$

Connecting these efforts with the positive directions of the nodal load (Figures 14.17, a, c), we obtain the relationship between the nodal load vector $\vec{F}^*$ and the vector $\vec{S}$ in the form:

$$\vec{F}^* = \begin{bmatrix} F_i^{\xi} \\ F_i^{\eta} \\ m_i \\ F_j^{\xi} \\ F_j^{\eta} \\ m_j \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1/l & 1/l \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1/l & -1/l \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} N \\ M_B \\ M_E \end{bmatrix} = a^* \, \vec{S}. \qquad (14.15)$$

The first three components of the vector $\vec{F}^*$ determine the load on the node at the beginning of the bar, and the next three – at the end of the bar.

Through a^*, the bar equilibrium matrix is denoted in the local coordinate system:

$$a^* = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1/l & 1/l \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1/l & -1/l \\ 0 & 0 & 1 \end{bmatrix}.$$

Upon transition to the general coordinate system, the equilibrium equations of the bar (14.15) are transformed.

Consider the problem of transforming the coordinates of the vector of nodal forces in the transition from a local coordinate system to a common one.

From the equations of projections of linear forces in the *i*-th node on the axis of the general coordinate system (Figure 14.18) it follows that:

$$F_i^x = F_i^\xi \cos\varphi - F_i^\eta \sin\varphi,$$

$$F_i^y = F_i^\xi \sin\varphi + F_i^\eta \cos\varphi.$$

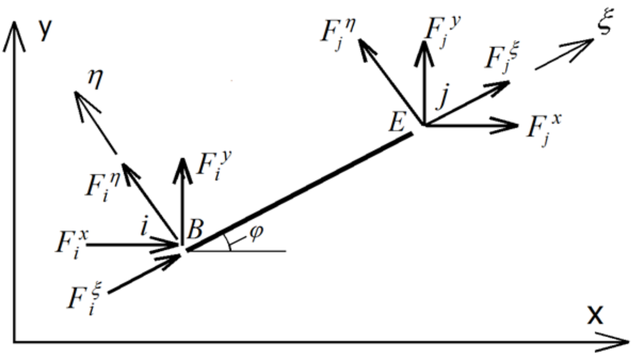


Figure 14.18

Given that the moment m_i remains unchanged when the coordinate system is rotated, we present the expression for the transformation of the forces of the *i*-th node in the form:

$$\vec{F}_i = \begin{bmatrix} F_i^x \\ F_i^y \\ m_i \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} F_i^\xi \\ F_i^\eta \\ m_i \end{bmatrix} = C^T \vec{F}_i^*, \qquad (14.16)$$

where C^T is matrix of the rotation operator when the vector is rotated through an angle clockwise.

Through C it is customary to denote the matrix of the operator of rotation of the vector counterclockwise.

Similar relations hold for forces in the *j*-th node:

$$\vec{F}_j = C^T \vec{F}_j^* .$$

Then, for the load vector in the nodes connected by the bar, the rotation transformation will be performed using the expression:

$$\vec{F} = \begin{bmatrix} C^T & 0 \\ 0 & C^T \end{bmatrix} \vec{F}^* = V^T \vec{F}^*, \qquad (14.17)$$

where

$$V^T = \left[\begin{array}{ccc|ccc} \cos\varphi & -\sin\varphi & 0 & 0 & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & 0 & 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

So, if equality (14.15) is multiplied on the left by the matrix V^T, then in the general coordinate system the vector of nodal forces $\vec{F}$ will be expressed through the vector of efforts $\vec{S}$ in the following form:

$$\vec{F} = a\vec{S}, \tag{14.18}$$

where a is a bar equilibrium matrix in the general coordinate system, i.e.:

$$a = V^T a^*$$

or

$$a = \left[\begin{array}{c|c|c} -\cos\varphi & \dfrac{\sin\varphi}{l} & -\dfrac{\sin\varphi}{l} \\ -\sin\varphi & -\dfrac{\cos\varphi}{l} & \dfrac{\cos\varphi}{l} \\ 0 & -1 & 0 \\ \hline \cos\varphi & -\dfrac{\sin\varphi}{l} & \dfrac{\sin\varphi}{l} \\ \sin\varphi & \dfrac{\cos\varphi}{l} & -\dfrac{\cos\varphi}{l} \\ 0 & 0 & 1 \end{array}\right].$$

In this form, the equilibrium matrix is written for the bar with both pinched ends. As follows from (14.18), equilibrium matrices for bars with other conditions of supporting the ends can be obtained from this one by deleting rows and columns corresponding to zero forces in the bar.

In particular, if the left end of the bar has a hinge support ($M_B = 0$), and the other end is pinched, the equilibrium matrix is obtained from the original by deleting the second column and the third row. For bars with different options of support fastenings, the equilibrium matrices in the general coordinate system are written in the Table 14.2.

Let us determine the relationship between the deformations of the bar and the displacements of its ends. We write the displacement vector in the general coordinate system for the rod rigidly fixed at the ends in the form:

$$\vec{z} = \left[z_B^x, z_B^y, \varphi_B, z_E^x, z_E^y, \varphi_{E,} \right]^T,$$

where, as before, the indices "B" and "E" denote the beginning and ending of the bar. The displacements of the bar ends are the displacements of the nodes that it connects.

Figure 14.19 shows the initial and deformed positions of the bar in the local coordinate system.

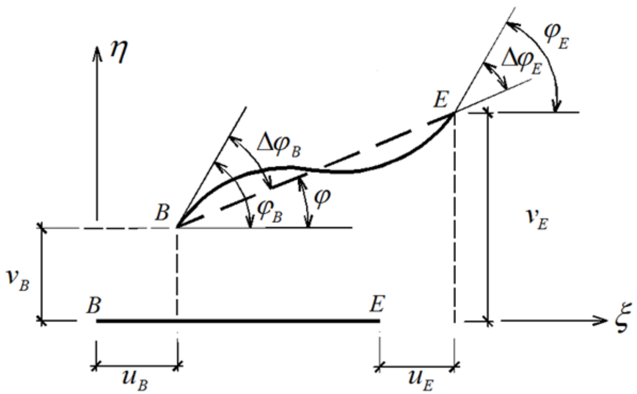


Figure 14.19

Table 14.2

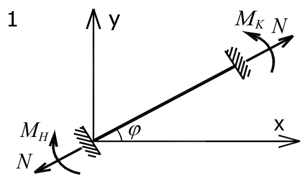
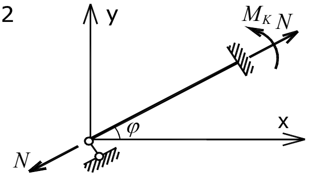
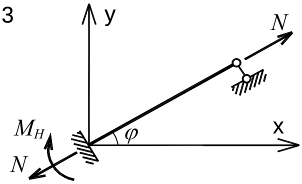
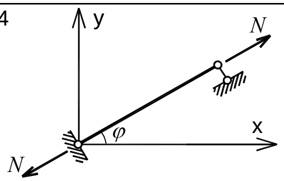
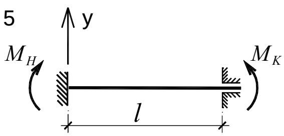
Option	Matrix α	Matrix k
1	2	3
1  $S^T = [N, M_B, M_E]$ $\Delta^T = [\Delta l, \Delta\varphi_B, \Delta\varphi_E]$	$\begin{bmatrix} -\cos\varphi & \frac{\sin\varphi}{l} & -\frac{\sin\varphi}{l} \\ -\sin\varphi & -\frac{\cos\varphi}{l} & \frac{\cos\varphi}{l} \\ 0 & -1 & 0 \\ - & - & - \\ \cos\varphi & -\frac{\sin\varphi}{l} & \frac{\sin\varphi}{l} \\ \sin\varphi & \frac{\cos\varphi}{l} & -\frac{\cos\varphi}{l} \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{EA}{l} & 0 & 0 \\ 0 & \frac{4EI}{l} & -\frac{2EI}{l} \\ 0 & -\frac{2EI}{l} & \frac{4EI}{l} \end{bmatrix}$

Table 14.2 (ending)

1	2	3
2  $S^T = [N, M_E]$ $\Delta^T = [\Delta l, \Delta\varphi_E]$	$\begin{bmatrix} -\cos\varphi & -\frac{\sin\varphi}{l} \\ -\sin\varphi & \frac{\cos\varphi}{l} \\ - & - \\ \cos\varphi & \frac{\sin\varphi}{l} \\ \sin\varphi & -\frac{\cos\varphi}{l} \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{EA}{l} & 0 \\ 0 & \frac{3EI}{l} \end{bmatrix}$
3  $S^T = [N, M_B]$ $\Delta^T = [\Delta l, \Delta\varphi_B]$	$\begin{bmatrix} -\cos\varphi & \frac{\sin\varphi}{l} \\ -\sin\varphi & -\frac{\cos\varphi}{l} \\ 0 & -1 \\ - & - \\ \cos\varphi & -\frac{\sin\varphi}{l} \\ \sin\varphi & \frac{\cos\varphi}{l} \end{bmatrix}$	$\begin{bmatrix} \frac{EA}{l} & 0 \\ 0 & \frac{3EI}{l} \end{bmatrix}$
4  $S = N \quad \Delta = \Delta l$	$\begin{bmatrix} -\cos\varphi \\ -\sin\varphi \\ - \\ \cos\varphi \\ \sin\varphi \end{bmatrix}$	$\begin{bmatrix} \frac{EA}{l} \end{bmatrix}$
5  $S^T = [M_B, M_E]$ $\Delta^T = [\Delta\varphi_B, \Delta\varphi_E]$	$\begin{bmatrix} -1/l & 1/l \\ -1 & 0 \\ - & - \\ 1/l & -1/l \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{4EI}{l} & -\frac{2EI}{l} \\ -\frac{2EI}{l} & \frac{4EI}{l} \end{bmatrix}$

The elongation of the bar and the angles of rotation of its end sections are components of the strain vector:

$$\vec{\delta} = \left[\Delta l,\ \Delta\varphi_B,\ \Delta\varphi_E\right]^T.$$

As follows from the Figure 14.19:

$$\Delta l = u_E - u_B,$$

$$\Delta\varphi_B = -(\varphi_B - \varphi) = -\varphi_B + \frac{\nu_E - \nu_B}{l},$$

$$\Delta\varphi_E = \varphi_E - \varphi = \varphi_E + \frac{\nu_E - \nu_B}{l}.$$

The direction of the rotation angle $\Delta\varphi_B$ does not coincide with the positive direction of the moment M_B, therefore the expression $(\varphi_B - \varphi)$ is accepted as negative. Using the matrix formula for writing, we get:

$$\vec{\delta} = \begin{bmatrix} \Delta l \\ \Delta\varphi_B \\ \Delta\varphi_E \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{l} & -1 & 0 & \frac{1}{l} & 0 \\ 0 & \frac{1}{l} & 0 & 0 & -\frac{1}{l} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_E \\ v_B \\ \varphi_B \\ u_E \\ v_E \\ \varphi_E \end{bmatrix} = a^{*\,T}\,\vec{z}^{\,*}, \quad (14.19)$$

where $\vec{z}^{\,*} = \left[u_B\ \ v_B\ \varphi_B\ u_E\ v_E\ \varphi_E\right]^T$ – is vector of displacements of the ends of the bar in the local coordinate system.

As in the case of operations with force vectors (14.17), the transformation of the coordinates of the vector $\vec{z}^{\,*}$ when the axes $\xi\,\eta$ are rotated by an angle φ clockwise is performed using the matrix V^T. Therefore, we can write that:

$$\vec{z} = V^T \vec{z}^{\,*}.$$

Consequently,

$$\vec{z}^{\,*} = V\,\vec{z}.$$

Then, in the general coordinate system, geometric equations, which are conditions of the compatibility of displacements of nodes (end sections of the bar) and deformations of the bar, can be written in the form:

$$\vec{\delta} = a^{*T} V\,\vec{z} = a^{T}\vec{z}. \tag{14.20}$$

We turn further to the physical equations, which describe the relationship of the deformation of the rod with the forces in it. Previously (section 14.7), it was shown that for a linearly deformable bar, this relationship is represented as $\vec{\Delta}_i = D_i\,\vec{S}_i$ (the index "i" corresponds to the number of the bar), or in expanded form for a bar with rigidly fixed ends, without entering its number of designation, in the form:

$$\vec{\delta} = \begin{bmatrix} \Delta l \\ \Delta\varphi_B \\ \Delta\varphi_E \end{bmatrix} = \begin{bmatrix} \frac{l}{EA} & 0 & 0 \\ 0 & \frac{2l}{6EJ} & \frac{l}{6EJ} \\ 0 & \frac{l}{6EJ} & \frac{2l}{6EJ} \end{bmatrix} \cdot \begin{bmatrix} N \\ M_B \\ M_E \end{bmatrix} = d\vec{S},$$

where d is the matrix of the internal compliance of the bar.

For bars with other conditions for joining nodes, physical dependencies are established using well-known methods for determining the end displacements. So, for a bar pivotally supported at the beginning and pinched at the end, the relationship $\vec{\delta}$ and $\vec{S}$ is obtained in the form:

$$\vec{\delta} = \begin{bmatrix} \Delta l \\ \Delta\varphi_E \end{bmatrix} = \begin{bmatrix} \frac{l}{EA} & 0 \\ 0 & \frac{l}{3EJ} \end{bmatrix} \cdot \begin{bmatrix} N \\ M_E \end{bmatrix} = d\vec{S}.$$

And for a bar with pinching at the beginning and a hinge support at the end:

$$\vec{\delta} = \begin{bmatrix} \Delta l \\ \Delta\varphi_B \end{bmatrix} = \begin{bmatrix} \dfrac{l}{EA} & 0 \\ 0 & \dfrac{l}{3EJ} \end{bmatrix} \cdot \begin{bmatrix} N \\ M_н \end{bmatrix} = d\vec{S}.$$

If it is necessary to write a physical law in the form $\vec{S} = S\left(\vec{\delta}\right)$, then from the presented expressions it follows that:

$$\vec{S} = d^{-1}\,\vec{\delta} = k\,\vec{\delta}, \tag{14.21}$$

where k is the matrix of internal stiffness (matrix of reactions) of the bar.

For example, for a bar with pinched ends:

$$k = \begin{bmatrix} \dfrac{EA}{l} & 0 & 0 \\ 0 & \dfrac{4EJ}{l} & -\dfrac{2EJ}{l} \\ 0 & -\dfrac{2EJ}{l} & \dfrac{4EJ}{l} \end{bmatrix}.$$

The equations of structural mechanics for a separate bar allow us to automate the process of forming the equilibrium matrix and the matrix of internal stiffness for an arbitrary planar bars system.

14.14. Forming the Equilibrium and Internal Stiffness Matrices for a Bars System

The format of the equilibrium matrix of the bars system is determined by the number and type of its nodes and elements. The number of rows of the matrix is equal to the number of equilibrium equations, that is, the number of degrees of freedom of the nodes. The number of columns is equal to the number of unknowns. Thus, the matrix (A) has dimensions $(m \cdot n)$.

The structure of the equilibrium matrix A can be represented in blocks form. For each rigid node, three lines are provided in which the coefficients from the equations $\sum X = 0$, $\sum Y = 0$ and $\sum M = 0$ are written sequentially. For each hinged node, two lines of coefficients are written from the equations $\sum X = 0$ and $\sum Y = 0$. In the orthogonal direction, vertical, the matrix appears to be divided into blocks columns, the number of which is equal to the number of bars. The width of the blocks column (the number of simple columns in it) is determined by the length of the vector $\vec{S}$ for each bar.

Let's compose the equilibrium matrix using the example of a two-span frame (Figure 14.20).

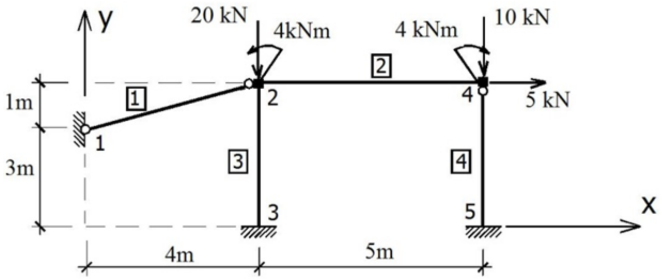


Figure 14.20

The equilibrium matrix A is written in table. 14.3. For easier orientation in the matrix structure, explanatory notes are given in the upper part of the table and to the left of it. The blocks of columns of the table correspond to bars of the frame. The number of columns in the block corresponds to the conditions of adjacency of the separate bar with the nodes of the frame. In each block of columns, the matrix a of an individual bar is located. The upper part of this matrix is connected with the beginning of the bar, and the lower – with its end.

Table 14.3

Bars			1	2			3			4	
S^T			N_1	N_2	M_{B2}	M_{E2}	N_3	M_{B3}	M_{E3}	N_4	M_{E4}
Nodes	*№№*		1	2	3	4	5	6	7	8	9
2	1		0.9701	-1	0	0	0	-0.25	0.25	0	0
	2		0.2425	0	-0.2	0.2	1	0	0	0	0
	3		0	0	-1	0	0	-1	0	0	0
4	4		0	1	0	0	0	0	0	0	0.25
	5		0	0	0.2	-0.2	0	0	0	1	0
	6		0	0	0	1	0	0	0	0	0

For example, the both ends of bar 2 are rigidly fixed at the nodes 2 and 4. Its matrix a has dimensions of 6x3.The first three lines refer to the beginning of the bar, that is, to node 2 (see the horizontal direction in the table), and the remaining lines to node 4.

If support links are imposed on one end of the bar (the bar is adjacent to the support node) and the forces in these links do not need to be calculated (no support reactions need to be determined), then the part of the bar equilibrium matrix a associated with the support node does not fit into the general equilibrium matrix of the A. So, the numerical values of the matrix a for the 3rd bar (5th, 6th and 7th columns) refer only to the 2nd node. A similar distribution of records takes place on the 1st and 4th bars.

The exclusion from the matrix of equations of equilibrium of the support nodes makes it possible to reduce the size of the matrix, which is expedient from a computational point of view..

For a deeper understanding of the physical meaning of the task, and also for the purpose of recording control during manual preparation of the initial data, one should sometimes check the record of individual equilibrium equations for the system nodes. So, for the same frame, the equation $\sum Y = 0$ for the 2-nd node (Figure 14.21) is written in the form:

$$0.2425 \cdot N_1 - 0.2 \cdot M_{B2} + 0.2 \cdot M_{E2} + N_3 = -20 \cdot 10^3 ,$$

where the substitution

$$Q_2 = \frac{M_{E2} - M_{B2}}{l_2}$$

has already been taken into account.

A similar structure of matrix A takes place for other systems (beams, arches, trusses, etc.).

So, the number of rows in matrix A is equal to the number of equilibrium equations of the system nodes, the number of columns is the number of unknown efforts (components of the vector $\vec{S}$ are indicated in the upper part of the table containing the equilibrium matrix).

Returning to the question of the degree of freedom of the system, we note that in our example $m = 6, n = 9$. The degree of static indeterminacy $k = n - m = 3$.

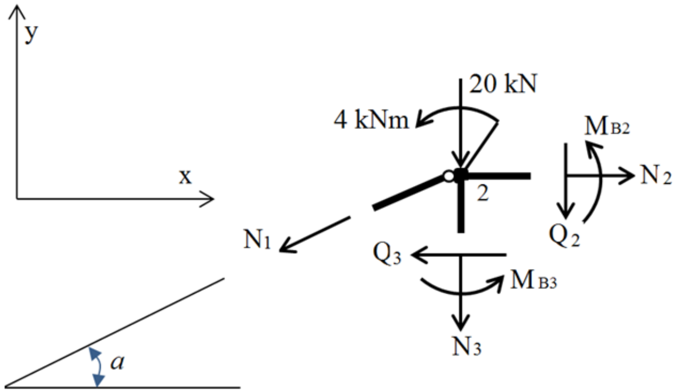


Figure 14.21

For a geometrically unchanged bars system, the rank of the equilibrium matrix is equal to the number of independent equilibrium equations for the nodes of this system, that is $r(A) = m.$ Moreover, if $m = n$ and the determinant of the equilibrium matrix $\det A \neq 0,$ then the investigated system is statically determinate; if $m < n,$ then the system is statically indeterminate.

When $m > n,$ the matrix rank can be $r(A) \leq n,$ but in any case the system is geometrically changeable.

As it has already been noted the number of components in the vectors $\vec{S}$ and $\vec{\Delta}$ is the same. In the table 14.3, the vectors $\vec{S}$ are recorded for each bar. The number of equilibrium equations, the coefficients of which are written in the matrix A, corresponds to the number of determined components of the displacement vector $\vec{z}$ both for an individual node and for the system as a whole.

The row number in the matrix A also indicates the number of the corresponding component of the vector $\vec{z}.$ For example, the second row $\left(\sum Y = 0\right)$ of the matrix A corresponds to the vertical displacement of node 2 and the second component in the displacement vector $\vec{z}.$ The total number of unknown displacements for the problem under consideration in the adopted formulation is six.

The matrix of internal stiffness of a single rod is square. Its size is determined by the number of components of the bar vector $\vec{S}.$ For the entire system, the internal stiffness matrix K has a quasi-diagonal structure; for the frame under consideration, it is presented in Table 14.4.

The indicated consistency of the matrices in the general equations of structural mechanics allows us to compose an algorithm for solving a mathematical model of the problem of verification calculation of bars systems.

Table 14.4

		1	2	3	4	5	6	7	8	9
1		14.82								
2			12.22	0	0					
3			0	0.8	-0.4					
4			0	-0.4	0.8					
5						15.28	0	0		
6						0	1	-0.5		
7						0	-0.5	1		
8									15.28	0
9									0	0.75
Note. All elements of the internal stiffness matrix have a factor of $30.24 \cdot 10^6$										

Automated calculation of the bars system involves the formation of matrices of general equations and the solution of the latter based on the initial data on the system, which include:

– the number of nodes, including support, and their signs (properties);
– coordinates of nodes;
– the location of the bars connecting the nodes;
– stiffness of the bars;
– information about the load acting on the nodes.

Since the elements of the matrix are the sines and cosines of the angles of inclination of the bars to the coordinate axes, their calculation is reduced to determining the quotient of dividing the lengths of the projections of the bars on the coordinate axis (the difference in the coordinates of the end and beginning of the bar) by the lengths of the bars.

Depending on the tasks, calculation results can be values of the node displacements, the forces in the bars and their deformations, the matrix of the external rigidity of the system, the matrix of the influence for forces and displacements. This information allows you to identify the features of the system under load and can be used to plot diagrams of efforts and displacements, lines of influence of efforts and displacements.

For the frame under consideration, we determine the displacements of the nodes, the internal forces in the bars, and construct the force diagrams.

If we accept

$$\vec{F} = [0; -20; 4; 5; -10; -4]^T \cdot 10^3,$$

where the dimension of concentrated forces and moments is *N* and *Nm*, then from the expression

$$\vec{z} = \left(A\,K\,A^{\mathrm{T}}\right)^{-1} \vec{F}$$

we find that:

$$z = \left[z_2^x,\, z_2^y,\, \varphi_2,\, z_3^x,\, z_3^y,\, \varphi_3\right] =$$

$$= [0.1910;\ -0.4348;\ 1.2104;\ 0.3251;\ -0.2332;\ -2.1982]^T \cdot 10^{-4}.$$

The sixth component of the vector $\vec{z}$ corresponds to the angle of rotation of the section at the end of the 2nd bar.

The effort vector is determined by the ratio

$$\vec{S} = K\,A^{\mathrm{T}}\,\vec{z}.$$

$$S = \left[3.58; \middle| 4.95;\ -0.12;\ -4.00; \middle|\right.$$

$$\left. -20.09; -3.88;\ 2.05; \middle| -10.78;\ 0.18\right]^T \cdot 10^3.$$

Vertical lines separate the effort components related to specific bars.

Diagrams of efforts in the frame are shown in Figure 14.22.

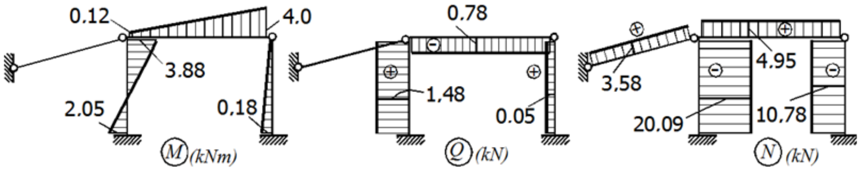


Figure 14.22

To determine the transverse forces in the bars, the dependence had been used:

$$Q = \frac{M_E - M_B}{l}.$$

Namely

$$Q_2 = \frac{-4.0 + 0.12}{5} = -0.776; \quad Q_3 = \frac{2.05 + 3.88}{4} = 1.4825;$$

$$Q_4 = \frac{0.18}{4} = 0.045.$$

E x a m p l e. Determine the forces in the rods of a statically indeterminate truss (Figure 14.23).

The rigidity of the rods are taken equal to:

$$EA_{1-4} = EA_{1-3} = EA_{2-4} = EA, \; EA_{1-2} = EA_{2-3} = EA_{3-4} = 2EA.$$

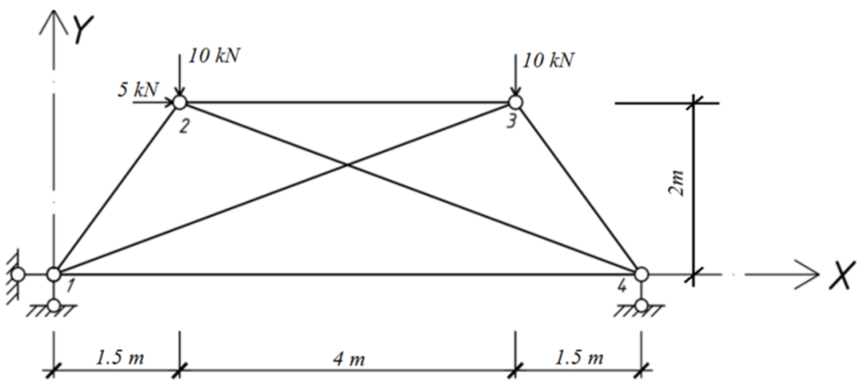


Figure 14.23

The number of unknown efforts $n = 6$. The degree of kinematic indeterminacy $m = 5$. Degree of static indeterminacy $k = n - m = 1$.

The calculation is performed by the displacement method using the general equations of structural mechanics. The primary system and the positive directions of the primary unknowns are shown in Figure 14.24.

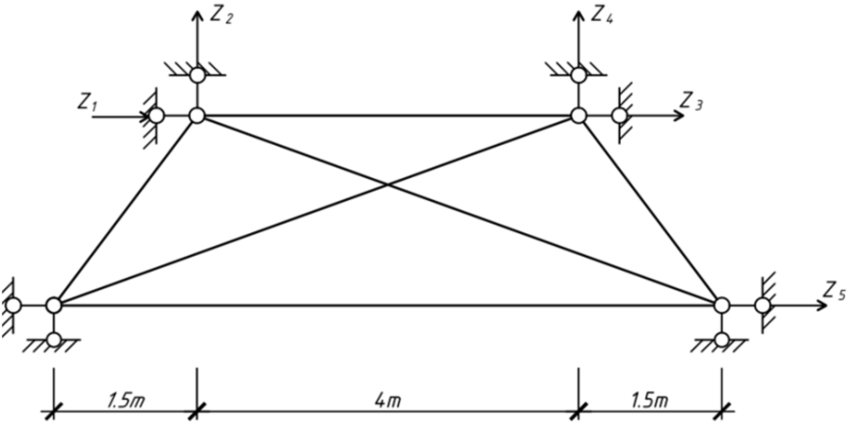


Figure 14.24

The equilibrium matrix A is formed “column by column” in accordance with the recommendations of section 14.14. It is shown in the table below.

Type of Equation	Nodes		Rods					
			1-4	2-3	1-2	3-4	1-3	2-4
$\sum X = 0$	2	A =		-1	0.6			-0.939793
$\sum Y = 0$					0.8			0.341743
$\sum X = 0$	3			1		-0.6	0.939793	
$\sum Y = 0$						0.8	0.341743	
$\sum X = 0$	4		**1**			0.6		0.939793

The matrix of internal rigidity K for the entire system is square, it has a diagonal structure:

$$diagK = \{1/7,\ 2/4,\ 2/2.5,\ 2/2.5,\ 1/\sqrt{34.25},\ 1/\sqrt{34.25}\} \cdot EA.$$

The external stiffness matrix is calculated by the expression:

$$R = A K A^T$$

and is given in the table below.

R =	0.938916	0.329122	-0.5	0	-0.150916	· EA
	0.329122	0.531956	0	0	0.0548784	
	-0.5	0	0.938916	-0.329122	-0.288	
	0	0	-0.329122	0.531956	0.384	
	-0.150916	0.0548784	-0.288	0.384	0.581773	

The system of equations of the displacement method is written as:

$$R\vec{z} = \vec{F}\ .$$

If det $R \neq 0$, then the external stiffness matrix R is a nonsingular matrix and can be inverted.

The calculated values of the elements of the inverse matrix R^{-1} are given in the table below.

R^{-1}=	4.29685	-2.98726	3.14043	-0.357711	3.18716	· 1/EA
	-2.98726	3.99269	-2.20723	0.485925	-2.56495	
	3.14043	-2.20723	3.67116	0.422316	2.56147	
	-0.357711	0.485925	0.422316	3.99269	-2.56495	
	3.18716	-2.56495	2.56147	-2.56495	5.74863	

Load $\vec{F}$ = [5, –10, 0, –10, 0]^T in the given truss causes node displacements the values of which can be determined with the matrix formula

$$\vec{z} = R^{-1}\vec{F}.$$

The calculated displacement values are also written in the table below.

Node numbers	2		3		4
Displacements Z*EA, m	54.934	-59.722	33.551	-46.575	67.235

Note. The first component of displacements for nodes 2 and 3 corresponds to the displacement along the X axis, and the second along the Y axis.

The vector of the truss rod forces $\vec{N}$ is calculated by expression:

$$\vec{N} = KA^T\vec{z}\ :$$

Rods	1-4	2-3	1-2	3-4	1-3	2-4
Forces, kN	9.605	-10.691	-11.854	-13.640	2.668	-1.512

The matrix R^{-1} can also be considered as a compliance matrix of the given truss, i.e., of the original truss without additional nodal links. The compliance matrix allows expressing displacements through loads:

$$\vec{z} = R^{-1}\vec{F} = D\vec{F}.$$

The compliance matrix of the truss under consideration (Figure 14.23) can be written as

$$R^{-1} = D = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} & \delta_{14} & \delta_{15} \\ \delta_{21} & \delta_{22} & \delta_{23} & \delta_{24} & \delta_{25} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \delta_{51} & \delta_{52} & \delta_{53} & \delta_{25} & \delta_{55} \end{bmatrix}.$$

If the given truss (Figure 14.23) is loaded by a unit force $F_1 = 1$ applied in the direction of the system displacement Z_1 (Figure 14.25), then the values of the all node displacements shown on the last figure will make up the first column of the compliance matrix $D = R^{-1}$.

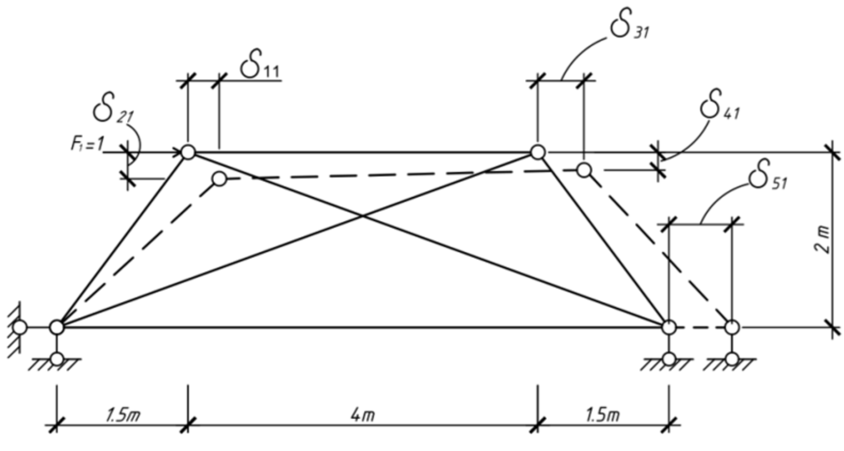


Figure 14.25

14.15. Influence Matrices for Displacements and Efforts

From the equality

$$D = R^{-1}$$

it follows that the displacement vector $\vec{z}$ can be expressed through the load vector $\vec{F}$ according to the formula

$$\vec{z} = D\vec{F}.$$

Therefore, the external compliance matrix D inverse to the external rigidity matrix R, is also an influence matrix of displacements L_z, i.e.

$$\vec{z} = D\vec{F} = L_z\vec{F} ,$$

where

$$L_z = D = R^{-1} = (AKA^T)^{-1} = [\delta_{ij}]$$

Its size is $(m \times m)$. Using this influence matrix L_z, the vector of nodal forces $\vec{F}$ is transformed into a vector of nodal displacements $\vec{z}$. The element δ_{ij} of this matrix determines the displacement of the system node in the i^{th} direction from the unit force $F_j = 1$.

The force vector $\vec{S}$ in a bars system can also be expressed in terms of the vector $\vec{F}$. For this purpose, we write it first in the form

$$\vec{S} = KA^T\vec{z} ,$$

and then, using the expression for $\vec{z}$, we represent it in the form

$$\vec{S} = K\ A^T\ R^{-1}\ \vec{F} = K\ A^T \left(A\ K\ A^T\right)^{-1} \vec{F} = L_S\ \vec{F},$$

where L_S is an influence matrix of efforts.

Its size is $(n \times m)$. Each element α_{ik} of this matrix determines the i^{th} internal force (i^{th} element in the vector $\vec{S}$) from the k^{th} external unit force ($F_k = 1$).

Elements α_{i1} of the first column of the matrix L_S are the internal forces in the all bars of the system due to the unit load $F_1 = 1$. Using these elements, you can plot the forces diagrams in the bars of the system from loading it by force $F_1 = 1$.

The elements of the first line show the values of the effort S_1 from the sequential loading of the system nodes by unit forces. Using these numbers, one can therefore construct a line of influence of the effort S_1. In this operation, from the first line it is necessary to select those numbers (elements) that correspond to a given direction of movement of a unit force.

Influence matrices are very important characteristics of the calculated (investigated) system. A change in a system of a parameter will necessarily entail a change in these matrices.

The physical meaning of the elements of the influence matrices also indicates that efforts diagrams or influence lines of efforts can be used to compile them. Such calculation methods are usually used for simple (with a small number of elements) systems. In other, more complex cases, it is advisable to apply the indicated mathematical formalization of this process using equilibrium matrices A and the internal stiffness matrices K.

Between influence matrices of efforts L_S and displacements L_z there is a relationship. Indeed, since

$$L_S = KA^T (AKA^T)^{-1} \text{ and } L_z = (AKA^T)^{-1},$$

then

$$L_S = KA^T L_z .$$

As for matrices of external rigidity (stiffness)

$$R = AKA^T$$

and external compliance

$$D = (AKA^T)^{-1} = R^{-1},$$

they are widely used in the dynamics and stability of structures.

E x a m p l e. We show the use of the general equations of structural mechanics for calculating a continuous beam (Figure 15.23, a). The beam is loaded with three load options and has a constant cross section:

$$A = 61.2 \cdot 10^{-4}\ \text{m}^2, \qquad I = 0.895 \cdot 10^{-4}\ \text{m}^4, \qquad E = 2.1 \cdot 10^5\ \text{MPa}.$$

Dimension of forces – kN, moments – kNm, lengths – m.

The total number of nodes is 9.

The boxes indicate the numbers of beam elements.

The matrix is formed "bar by bar" using the 5th option of the Table 14.2.

The internal stiffness matrix is quasi-diagonal:

$$diagK = [K_1 K_1 K_1 K_1 \mid K_2 K_2 K_2 K_2] \times 18.795 \cdot 10^3,$$

Where

$$K_1 = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 4/1.5 & -2/1.5 \\ -2/1.5 & 4/1.5 \end{bmatrix}.$$

The solution of the system

$$AKA^T z = F$$

gives a displacement matrix z. Linear displacements are measured in meters, turning angles are in radians.

The displacements diagram (Figure 14.26, b) corresponds to the first loading of the beam.

The efforts matrix S was calculated by the expression

$$S = KA^T z\ .$$

With its help, diagrams of bending moments were plotted for each load option (Figures 14.26, c, d, e).

To construct the lines of the influence of efforts, we used the matrix of the influence of internal forces:

$$L_S = KA^T (AKA^T)^{-1}.$$

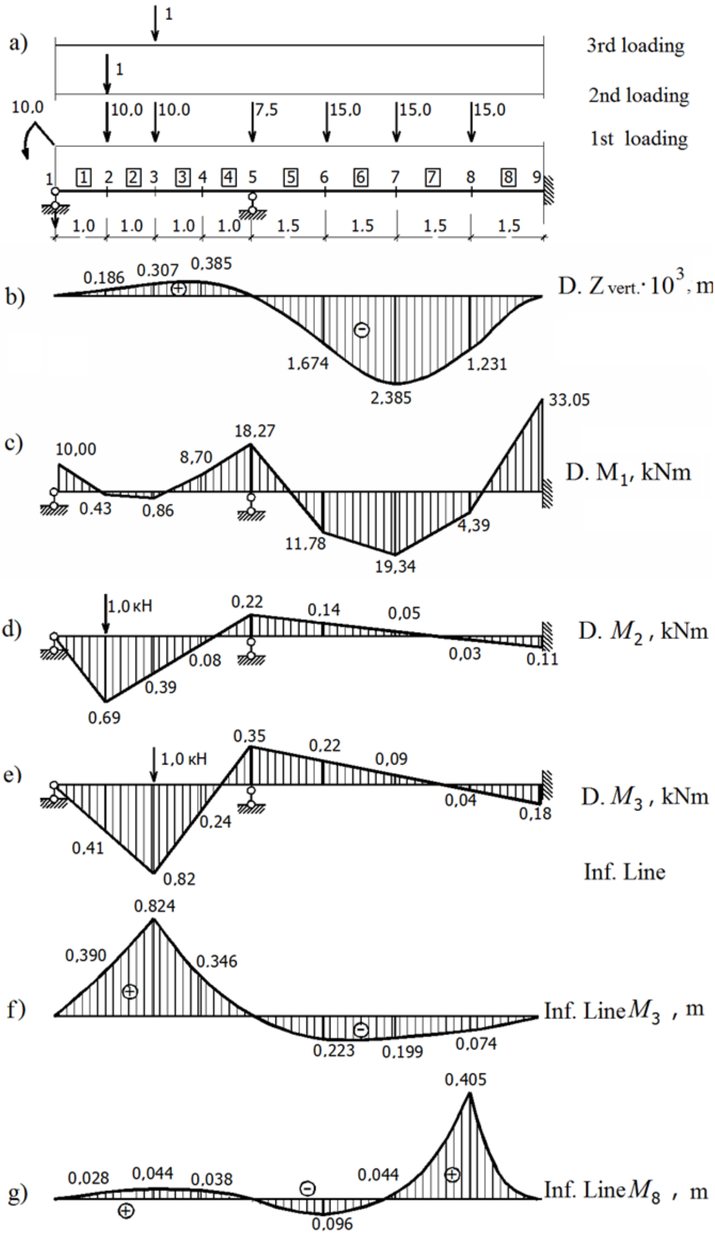

a)
3rd loading
2nd loading
1st loading
10,0
10,0
10.0
7.5
15,0
15,0
15,0
1 2 3 4 5 6 7 8 9
1.0 1.0 1.0 1.0 1.5 1.5 1.5 1.5
b)
0,186 0,307 0,385
1,674
2,385
1,231
D. Z vert.· 10^3, m
c)
10,00
8,70
18,27
33,05
0.43
0.86
11,78
19,34
4,39
D. M1, kNm
d)
1,0 кН
0,22 0,14 0,05
0,08 0,39 0,69
0,03 0,11
D. M2 , kNm
e)
1,0 кН
0,35 0,22 0,09
0,41 0,82 0,24
0,04 0,18
D. M3 , kNm
Inf. Line
f)
0.824
0,390
0.346
0,223 0,199 0,074
Inf. Line M3 , m
g)
0,028 0,044 0,038
0,096
0,044
0,405
Inf. Line M8 , m

Figure 14.26

The elements in columns Nos. 2, 4, 6, 9, 11, 13 are the efforts in the beam caused by the vertical concentrated unit force applied, respectively, at points Nos. 2, 3, 4, 6, 7, 8. Columns Nos. 1, 3, 5, 7, 8, 10, 12, 14 contain information about the efforts due to the concentrated unit moment applied, respectively, at points Nos. 1, 2, 3, 4, 5, 6, 7, 8.

The rows of the matrix L_S contain the ordinates of the influence lines for internal forces in the corresponding cross sections of the beam. So, with the help of the elements of the 5th row, the $Inf.Line\, M_3$ was built (figure 14.26, f). The effort M_3 is the bending moment in the cross section No. 3 (at point No. 3). This moment can be considered as a bending moment M_{B3} at the beginning of the third element, or as a bending moment M_{E2} at the end of the second element of the beam. The ordinates of M_{E2} are in the 4th row of L_S in columns Nos. 2, 4, 6, 9, 11, 13.

To build $Inf.Line\, M_8$ (Figure 14.26, g) the ordinates from the 15th row of L_s were used.

14.16. Spatial Trusses

The coordinates of nodes of the spatial truss will be considered known. For rod P_1P_2 (P_1 is node at the beginning, P_2 is node at the end of the rod), as for the directed segment $\overrightarrow{P_1P_2}$, we find the direction cosines $\cos\alpha_x$, $\cos\alpha_y$, $\cos\alpha_z$ by the expressions:

$$\cos\alpha_x = \frac{x_2 - x_1}{l}, \quad \cos\alpha_y = \frac{y_2 - y_1}{l}, \quad \cos\alpha_z = \frac{z_2 - z_1}{l},$$

where

$$l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The direction cosines of the directed segment $\overrightarrow{P_2P_1}$ are

$$-\cos\alpha_x,\ -\cos\alpha_y,\ -\cos\alpha_z .$$

We will consider the tensile longitudinal force in the rod to be positive. In the terminal sections of the rod, it has opposite directions. These directions correspond to the directions of segments $\overrightarrow{P_2P_1}$ and $\overrightarrow{P_1P_2}$.

The projections of the directed segment $\overrightarrow{P_1P_2}$ on axis OX, OY, OZ are equal respectively:

$$l_x = x_2 - x_1 = l\cos\alpha_x, \quad l_y = y_2 - y_1 = l\cos\alpha_y,$$

$$l_z = z_2 - z_1 = l\cos\alpha_z .$$

Consequently, the projections of the longitudinal force N applied at point P_2 on the same axis are equal

$$N\cos\alpha_x,\ N\cos\alpha_y, N\cos\alpha_z,$$

and the projections of the force N applied at the point P_1 must be recorded with the opposite sign:

$$-N\cos\alpha_x,\ -N\cos\alpha_y, -N\cos\alpha_z .$$

The numbering of truss nodes determines the numbering of nodes P_1 and P_2, connected by a rod. Moreover, for the beginning of the rod, that is, for point P_1, a node with a lower number is taken.

In the equilibrium matrix A, with each free node of the spatial truss, three rows are connected, in which the coefficients are written for the forces N in the rods adjacent to this node. These coefficients are factors (direction cosines) with N in the equations:

$$\sum X = 0, \quad \sum Y = 0, \quad \sum Z = 0.$$

In practical problems, it is more convenient to form a matrix A not in rows but in columns. Recall once again that the direction cosines of the rod in the equations related to the nodes P_1 and P_2 will have opposite signs. When forming a matrix by columns, it is recommended to use a template vector $\vec{a}$ for each bar:

$$\vec{a} = [-\cos\alpha_x,\ -\cos\alpha_y, -\cos\alpha_z, \quad \cos\alpha_x,\ \cos\alpha_y, \cos\alpha_z]^T .$$

The first three elements of the vector refer to the beginning of the rod (node P_1), the remaining three to the end of it (node P_2).

The general equations of structural mechanics for a spatial truss have the same notation as for a plane system.

N o t e. For plane trusses the equilibrium matrix can also be formed through the direction cosines using the above vector $\vec{a}$, excluding the components in it: $-\cos\alpha_z, ..., \cos\alpha_z$.

E x a m p l e. Determine the forces in the rods of the spatial truss (Figure 14.27). The rigidity of all rods is taken equal to EA. Load vector is $\vec{F} = [5.0; 10.0; -4.0]^T$ kN.

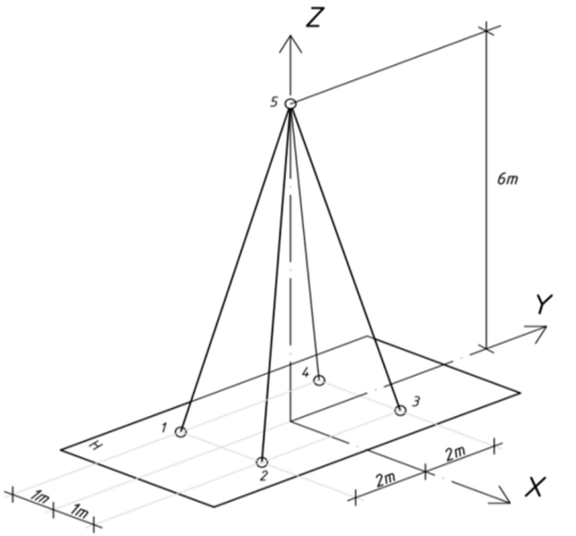


Figure 14.27

Having determined the direction cosines of the rods, we form the equilibrium matrix:

	E. 1–5	E. 2–5	E. 3–5	E. 4–5
	+0.15617	-0.15617	- 0.15617	0.15617
A =	+0.31235	+0.31235	- 0.31235	- 0.31235
	+0.93704	+0.93704	+0.93704	+0.93704

The truss internal stiffness matrix is diagonal:

$$diagK = [0.15617, 0.15617, 0.15617, 0.15617]EA\,.$$

The matrix of the external rigidity of the truss is obtained by the formula $R = AKA^T$ and has the form:

R=

0.0152365	0	0
0	0.0609461	0
0	0	0.548514

· EA.

The matrix inverse to matrix R is shown below:

R^{-1} =

65.6317	0	0
0	16.4079	0
0	0	1.8231

1/EA.

The vectors of displacements of node 5 and the internal forces in the rods are calculated by known formulas.

$\vec{Z}$ = [328.159; 164.079; –7.292]^{T} 1/EA.
$\vec{N}$ = [14.941; –1.067; –17.075; –1.067]^{T}, kN.

E x a m p l e. Determine the forces in the rods of the spatial truss (Figure 14.28). The stiffness of all rods taken equal to EA.

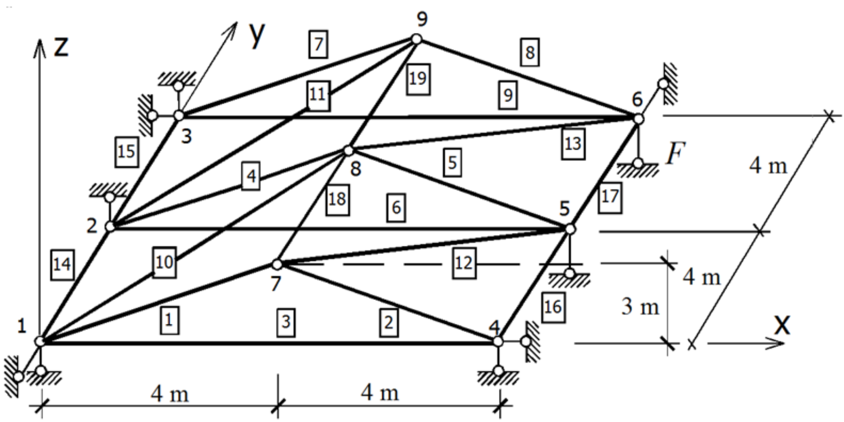


Figure 14.28

The load vector is taken in the form:

$$\vec{F} = [0;\, 0;\, 0;\, 0;\, 0;\, 0;\, 0;\, 0;\, –5.0;\, 5.0;\, –50.0;$$
$$–5.0;\, 5.0;\, –100.0;\, –5.0;\, 5.0;\, –50.0\,]^T\, kN.$$

The truss equilibrium matrix is given in table. 14.5.

In this example, the task of determining support reactions was not posed. Therefore, there are no rows in the matrix A corresponding to the equations of the projections of forces in the rods adjacent to the support nodes in the directions of the support links.

The truss stiffness matrix K is diagonal:

diagK= {0.2; 0.2; 0.125; 0.2; 0.2; 0.125; 0.2; 0.2; 0.125; 0.15617; 0.15617; 0.15617; 0.15617; 0.25; 0.25; 0.25; 0.25; 0.25; 0.25} EA.

The solution of the system of equations $\vec{R}z = \vec{F}$ gives:

$$z^T = [-382.96, -254.51, 14.27, 14.27, 25.73, 173.99,$$
$$25.73, 382.96, -236.14, 114.66, -539.03, -79.32,$$
$$120.39, -826.69, 158.01, 126.12, -621.12]\frac{1}{EA}\, m.$$

The efforts in the truss rods are determined by the expression:

$$\vec{N} = K\ A^T\ \vec{z}\ .$$

$$\vec{N}^T = [-44.79, -30.50, 47.87, -71.41, -58.91, 53.56, -49.25,$$
$$-38.54, 47.87, -19, 27, 5.71, -10.30, -27.27, 3.57, 0, 0, -6.43,$$
$$1.43, 1.43]\, kN.$$

Table 14.5

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	–0.8		–1							–0.6247									
2				–0.8		–1					–0.6247								
3											–0.6247			1	–1				
4															1				
5																–1			
6					0.8	1						0.6247							
7												0.6247				1	–1		
8								0.8	1				0.6247						
9	0.8	–0.8										–0.6247							
10												–0.6247						–1	
11	0.6	0.6										0.4685							
12				0.8	–0.8					0.6247			–0.6247						
13										0.6247			–0.6247					1	–1
14				0.6	0.6					0.4685			0.4685						
15							0.8	–0.8			0.6247								
16											0.6247								1
17							0.6	0.6			0.4685								

14.17. Spatial Frames

For each bar of the frame, the orientation of the axes of the local coordinate system (hereinafter referred to in small letters x, y, z) will be considered known. The axis ox is directed from node P_1 to node P_2 (from a node with a lower number to a node with a higher number). The axes oy and oz of the right Cartesian coordinate system are located in a plane perpendicular to the axis ox and passing through the point P_1. Since the position of the cross section of the frame bar is taken to be predetermined, the position of the axes oy and oz is also established. The location of the axes along each bar must be fixed unambiguously.

Let relative to the axes of the global coordinate system $OXYZ$:

– the axis ox has direction cosines t_{11}, t_{21}, t_{31};

– the axis oy has direction cosines t_{12}, t_{22}, t_{32};

– the axis oz has direction cosines t_{13}, t_{23}, t_{33}.

Then the local coordinate system adopted for the bar is characterized by a matrix of direction cosines:

$$T = \begin{bmatrix} t_{11} & t_{21} & t_{31} \\ t_{12} & t_{22} & t_{32} \\ t_{13} & t_{23} & t_{33} \end{bmatrix}.$$

Using the matrix T, Cartesian rectangular coordinates are transformed when the axes are rotated.

Define the force and strain vectors in the bar of the spatial frame in the following form:

$$\vec{S} = [N, M_T, M_{yb}, M_{ye}, M_{zb}, M_{ze}]^T,$$

$$\vec{\Delta} = [\Delta l, \Delta\varphi_T, \Delta\varphi_{yb}, \Delta\varphi_{ye}, \Delta\varphi_{zb}, \Delta\varphi_{ze}]^T$$

The efforts in the bar end sections, oriented along the axes of the local coordinate system, are expressed by the vector $\vec{r}^*$ through the vector $\vec{S}$ using the equilibrium matrix a^*:

$$\vec{r}^* = a^* \vec{S} ,$$

where

$$\vec{r}^* = [r_{xb}, r_{yb}, r_{zb}, m_{xb}, m_{yb}, m_{zb}, r_{xe}, r_{ye}, r_{ze}, m_{xe}, m_{ye}, m_{ze}]^T .$$

The components of the vector $\vec{r}^*$ are shown in Figure 14.29. The positive components directions of the vector $\vec{S}$ are shown in Figure 14.30.

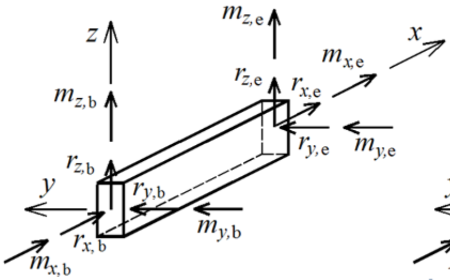


Figure 14.29

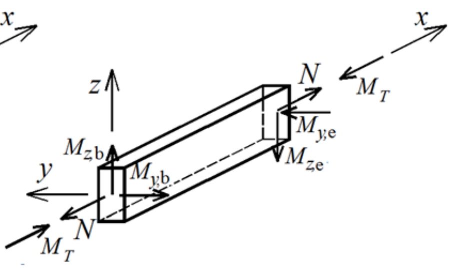


Figure 14.30

In these figures, a vector image of moments was used. The moment acting in a clockwise direction along a certain axis (when viewed from a point corresponding to the end of the coordinate axis) is depicted by a vector directed in the positive direction of the axis.

In Figure 14.29 notations accepted:

m_{xb}, m_{xe} are torques at the beginning and at the end of the bar;

m_{yb}, m_{ye} are bending moments at the beginning and at the end of the bar relative to the axis y;

m_{zb}, m_{ze} – bending moments at the beginning and at the end of the bar relative to the axis z.

The equilibrium conditions for the bar allow us to obtain the following relationships:

$$r_{xb} = -N; \qquad r_{xe} = -r_{xb};$$

$$r_{yb} = \frac{M_{ze} - M_{zb}}{l}; \qquad r_{ye} = -r_{yb};$$

$$r_{zb} = \frac{M_{ye} - M_{yb}}{l}; \qquad r_{ze} = -r_{zb};$$

$$m_{xb} = M_T; \qquad m_{xe} = -m_{xb};$$

$$m_{yb} = -M_{yb}; \qquad m_{ye} = -m_{yb};$$

$$m_{zb} = M_{zb}; \qquad m_{ze} = -m_{zb}.$$

The positive directions of the end forces (Figure 14.29) coincide with the directions of the axes of the local coordinate system. Therefore, to project these efforts on the axis of the global coordinate system, we use the matrix T:

$$R_{X,B} = t_{11} \cdot r_{x,b} + t_{12} \cdot r_{y,b} + t_{13} \cdot r_{z,b};$$

$$R_{Y,B} = t_{21} \cdot r_{x,b} + t_{22} \cdot r_{y,b} + t_{23} \cdot r_{z,b};$$

$$R_{Z,B} = t_{31} \cdot r_{x,b} + t_{32} \cdot r_{y,b} + t_{33} \cdot r_{z,b};$$

$$M_{X,B} = t_{11} \cdot m_{x,b} + t_{12} \cdot m_{y,b} + t_{13} \cdot m_{z,b};$$

$$M_{Y,B} = t_{21} \cdot m_{x,b} + t_{22} \cdot m_{y,b} + t_{23} \cdot m_{z,b};$$

$$M_{Z,H} = t_{31} \cdot m_{x,н} + t_{32} \cdot m_{y,н} + t_{33} \cdot m_{z,н}.$$

In matrix form, these expressions are represented as follows:

$$\vec{R}_b = T^T \, \vec{r}_b, \qquad \vec{M}_b = T^T \, \vec{m}_b^*,$$

where

$$\vec{R}_b = \left[R_{x,b}; \quad R_{y,b}; \quad R_{z,b} \right]^T;$$

$$\vec{M}_b = \left[M_{x,b}; \quad M_{y,b}; \quad M_{z,b} \right]^T;$$

$$\vec{r}_b^{\,*} = \left[r_{x,b}; \quad r_{y,b}; \quad r_{z,b} \right]^T; \qquad \vec{m}_b^{\,*} = \left[m_{x,b}; \quad m_{y,b}; \quad m_{z,b} \right]^T.$$

Similar relations hold for the efforts at the end of the bar:

$$\vec{R}_e = T^T \, \vec{r}_e^{\,*}, \qquad \vec{M}_e = T^T \, \vec{m}_e^{\,*},$$

where

$$\vec{R}_e = \left[R_{x,e}; \quad R_{y,e}; \quad R_{z,e} \right]^T;$$

$$\vec{M}_e = \left[M_{x,e}; \quad M_{y,e}; \quad M_{z,e} \right]^T;$$

$$\vec{r}_e^{\,*} = \left[r_{x,e}; \quad r_{y,e}; \quad r_{z,e} \right]^T; \quad \vec{m}_e^{\,*} = \left[m_{x,e}; \quad m_{y,e}; \quad m_{z,e} \right]^T.$$

Based on the recorded expressions, the reactions vector at the bar ends in the global coordinate system is defined as follows:

$$\vec{R} = a\,\vec{S},$$

where

$$\vec{R} = \left[R_{x,b}; \quad R_{y,b}; \quad R_{z,b}; \quad M_{x,b}; \quad M_{y,b}; \quad M_{z,b}; \right.$$
$$\left. R_{x,e}; \quad R_{y,e}; \quad R_{z,e}; \quad M_{x,e}; \quad M_{y,e}; \quad M_{z,e}; \right]^T;$$

a is bar equilibrium matrix in the general coordinate system:

$$a = \begin{array}{|c|c|c|c|c|c|}
\hline
-t_{11} & & -\frac{t_{13}}{l} & \frac{t_{13}}{l} & -\frac{t_{12}}{l} & \frac{t_{12}}{l} \\
\hline
-t_{21} & & -\frac{t_{23}}{l} & \frac{t_{23}}{l} & -\frac{t_{22}}{l} & \frac{t_{22}}{l} \\
\hline
-t_{31} & & -\frac{t_{33}}{l} & \frac{t_{33}}{l} & -\frac{t_{32}}{l} & \frac{t_{32}}{l} \\
\hline
 & t_{11} & -t_{12} & & t_{13} & \\
\hline
 & t_{21} & -t_{22} & & t_{23} & \\
\hline
 & t_{31} & -t_{32} & & t_{33} & \\
\hline
t_{11} & & \frac{t_{13}}{l} & -\frac{t_{13}}{l} & \frac{t_{12}}{l} & -\frac{t_{12}}{l} \\
\hline
t_{21} & & \frac{t_{23}}{l} & -\frac{t_{23}}{l} & \frac{t_{22}}{l} & -\frac{t_{22}}{l} \\
\hline
t_{31} & & \frac{t_{33}}{l} & -\frac{t_{33}}{l} & \frac{t_{32}}{l} & -\frac{t_{32}}{l} \\
\hline
 & -t_{11} & & t_{12} & & -t_{13} \\
\hline
 & -t_{21} & & t_{22} & & -t_{23} \\
\hline
 & -t_{31} & & t_{32} & & -t_{33} \\
\hline
\end{array}\,. \qquad (14.22)$$

The equilibrium matrices of rods and bars of plane and spatial trusses, plane frames, systems of cross beams are obtained as special cases from the written matrix a (14.22) by deleting the corresponding rows and columns.

The matrix of internal stiffness of the bar rigidly fixed at the ends has the following form:

$$K = \begin{array}{|c|c|c|c|c|c|} \hline \frac{EA}{l} & & & & & \\ \hline & \frac{GI_{KP}}{l} & & & & \\ \hline & & \frac{4EJ_y}{l} & -\frac{2EJ_y}{l} & & \\ \hline & & -\frac{2EJ_y}{l} & \frac{4EJ_y}{l} & & \\ \hline & & & & \frac{4EJ_z}{l} & -\frac{2EJ_z}{l} \\ \hline & & & & -\frac{2EJ_z}{l} & \frac{4EJ_z}{l} \\ \hline \end{array}.$$

The basic equations of structural mechanics for calculating the bars systems in the form of a displacement method are presented in the form:

$$R\,\vec{z} = \vec{F} + A\,K\,\vec{\Delta}'.$$

The matrix A of equilibrium equations of the calculated system is compiled element by element using the equilibrium matrices a (14.22) of the bars.

E x a m p l e. Plot the diagram of the longitudinal forces, torsional and bending moments in the frame (Figure 14.31). Accept the following stiffness ratios for all bars:

$$EAh^2 = GJ_T = EJ_y = EJ_z, \quad (h = 1m).$$

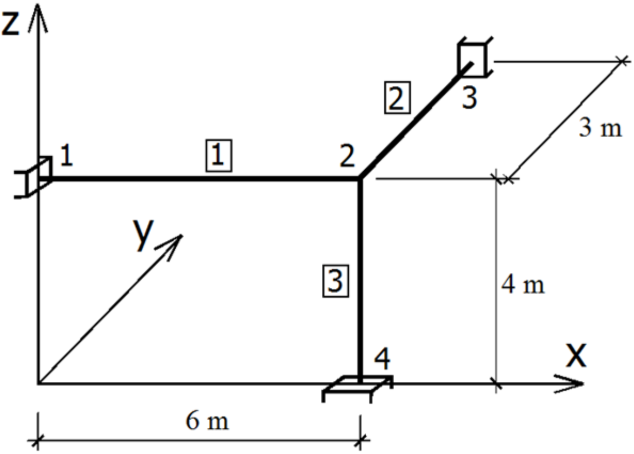


Figure 14.31

The positions of the local coordinate axes for each bar of the frame shown in Figure 14.32, determine the matrix of direction cosines:

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$$

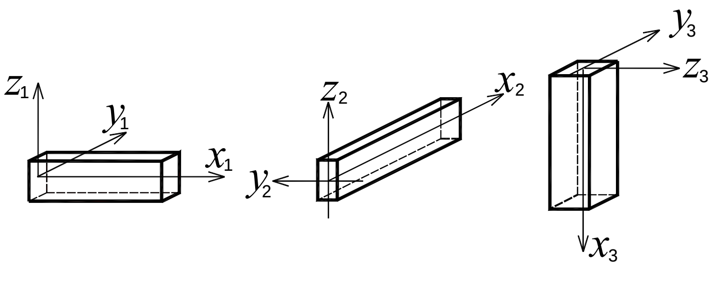


Figure 14.32

The matrix of frame equilibrium equations is given in table 14.6.

Table 14.6

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1										1/3	-1/3			-1/4	1/4		
2					1/6	-1/6	-1										-1/4	1/4
3			1/6	-1/6					-1/3	1/3			1					
4		-1							1								1	
5				1				1							-1			
6						-1					1			-1				

The matrix of the internal rigidity of the frame is quasi-diagonal:

$$diag\, K = \left[K_1, \quad K_2, \quad K_3\right] \cdot EJ_y,$$

where

$$K_1 = \begin{bmatrix} 1/6 & & & & & \\ & 1/6 & & & & \\ & & 4/6 & -2/6 & & \\ & & -2/6 & 4/6 & & \\ & & & & 4/6 & -2/6 \\ & & & & -2/6 & 4/6 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 1/3 & & & & & \\ & 1/3 & & & & \\ & & 4/3 & -2/3 & & \\ & & -2/3 & 4/3 & & \\ & & & & 4/3 & -2/3 \\ & & & & -2/3 & 4/3 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 1/4 & & & & & \\ & 1/4 & & & & \\ & & 1 & -0.5 & & \\ & & -0.5 & 1 & & \\ & & & & 1 & -0.5 \\ & & & & -0.5 & 1 \end{bmatrix}.$$

Taking a load vector

$$\vec{F}_2 = [0; \quad 0; \quad -11; \quad 3; \quad 7.5; \quad 0]^T,$$

where the dimension of forces in kN, moments in kNm, lengths in m, positive moments are directed relative to the axes of the general coordinate system in a clockwise direction, when viewed from a point corresponding to the end of the axis, we obtain:

$$\vec{z} = [-1.87; -2.75; -17.54; -3.89; 2.64; 0.76]^T \frac{1}{EJ_y},$$

$$\vec{S} = [-0.31; 0.65; -3.80; 4.68; -0.21; -0.05; | \, 0.92; 0.88; 6.51; -9.10; \\ -0.24; 0.74; | -4.38; -0.19; -1.94; 0.62; -2.86; 0.91;]^T.$$

The corresponding diagtams of efforts are shown in Figure 14.33. Figure 14.34 shows (in axonometric view) the bending moments and torques acting on the cut-out node 2. The moments are divided into groups according to their location with respect to the coordinate planes.

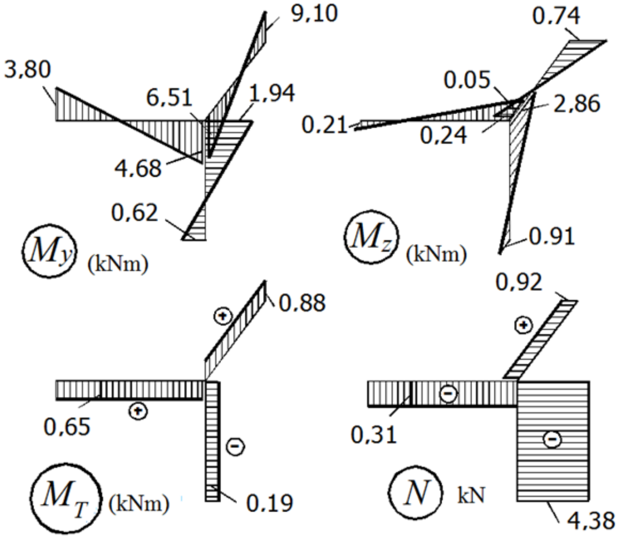
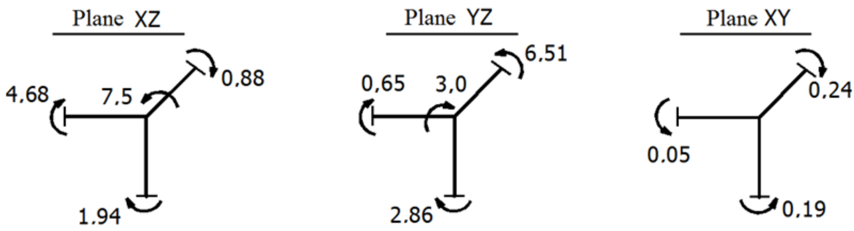


Figure 14.33



$\sum M_2 =$ $4.68 + 0.88 + 1.94 - 7.5 = 0.$ $\sum M_2 =$ $0.65 - 6.51 + 2.86 + 3 = 0.$ $\sum M_2 =$ $-0.05 - 0.19 + 0.24 = 0.$

Figure 14.34

For this node, the equations of projections of forces on the coordinate axes are also satisfied.

E x a m p l e. Construct efforts diagrams in the frame (Figure 14.35), taking for all bars

$$EA\,h^2 = 10EJ_y,\;\; GT = 0.27EJ_y,\;\; EJ_z = 0.5EJ_y\,.$$

The frame is loaded with two generalized nodal forces:

$$\vec{F}_1 = [12.0, 0, -98.0, 40.0, -52.0, 0]^T\,,$$

$$\vec{F}_2 = [0, 0, -98.0, 40.0, 64.0, 0]^T\,.$$

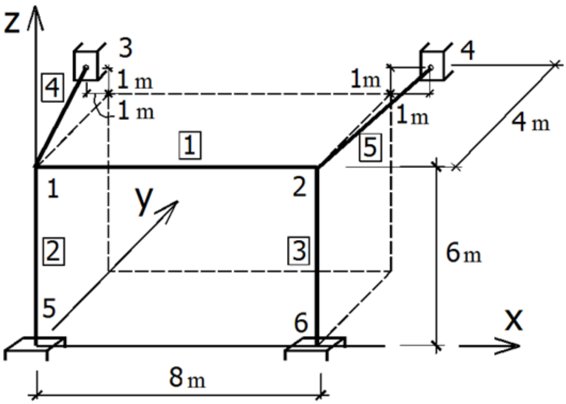


Figure 14.35

Dimension of forces – kN, moments – kNm, lengths – m.

The matrixes of the direction cosines of the axes of the local coordinate system for the frame bars are presented in the following forms:

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, T_2 = T_3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$T_4 = \begin{bmatrix} -2.35702 & 9.42809 & 2.35702 \\ -9.70143 & -2.42536 & 0 \\ 0.57166 & -2.28665 & 9.71825 \end{bmatrix} \cdot 10^{-1}$$

$$T_5 = \begin{bmatrix} 2.35702 & 9.42809 & 2.35702 \\ -9.70143 & 2.42536 & 0 \\ 0.57166 & -2.28665 & 9.71825 \end{bmatrix} \cdot 10^{-1}.$$

Matrices of internal rigidity of the 4th and 5th bars coincide:

$$K_4 = K_5 = \begin{bmatrix} 2.357 & & \\ & 0.707 & \\ & & 0.354 \end{bmatrix} \cdot EJ_y$$

After the formation of the matrix of external rigidity

$$R = AKA^T,$$

we solve the system of equations

$$R\vec{z} = \vec{F}$$

and determine the forces in the frame rods by expression

$$\vec{S} = KA^T\vec{z}.$$

Plots of efforts are shown in Figure 14.36.

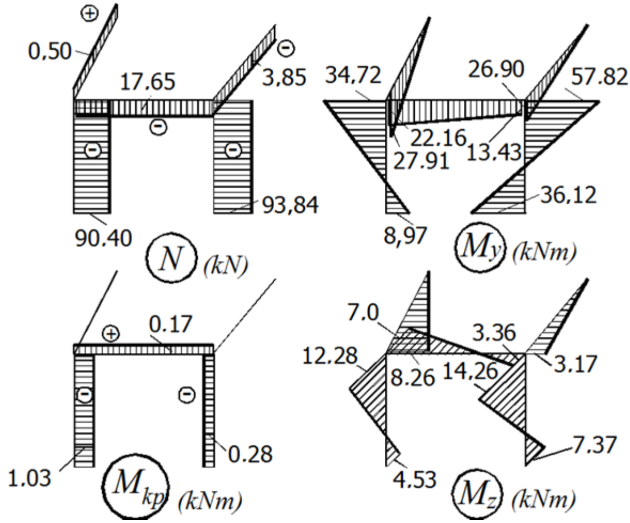


Figure 14.36

E x a m p l e. For the cross-beam system (Figure 14.37) we take the bending rigidities for all the bars equal to EJ and $GJ_T = 0.27EJ$.

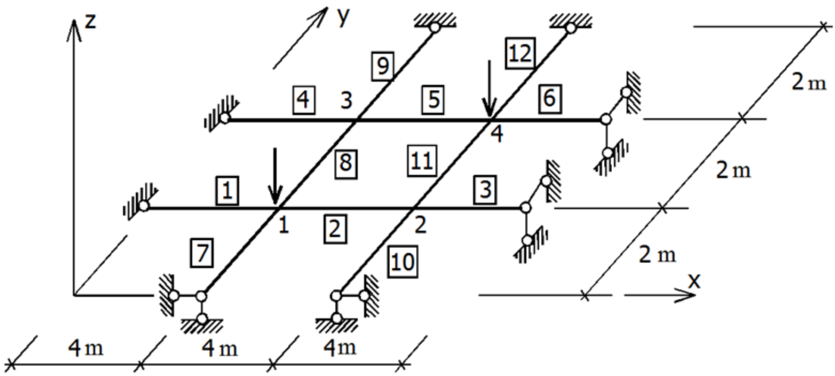


Figure 14.37

The matrix of equilibrium equations, compiled using the equilibrium matrix of the bar a in the general coordinate system, is written in Table 14.7.

Table 14.7

Bars		1	2			3	4	5			6	7	8			9	10	11			12
Efforts		Me	Mт	Mв	Me	Mв	Me	Mт	Mв	Me	Mв	Me	Mт	Mв	Me	Mв	Me	Mт	Mв	Me	Me
Numbering		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Node 1	ΣZ	–0.25		–0.25	–0.25							–0.5		–0.5	0.5						
	ΣM_x		1									–1		1							
	ΣM_y	1		–1									1								
Node 2	ΣZ			–0.25	–0.25	–0.25											–0.5		–0.5	0.5	
	ΣM_x		–1		1	–1											–1		1		
	ΣM_y																	1			
Node 3	ΣZ						–0.25		–0.25	0.25				0.5	–0.5	–0.5					
	ΣM_x							1							–1	1					
	ΣM_y						1		–1				–1								
Node 4	ΣZ								0.25	–0.25	–0.25								0.5	–0.5	–0.5
	ΣM_x							–1												–1	1
	ΣM_y									1	–1							–1			

THEME 15. VARIATIONAL PRINCIPLES AND VARIATIONAL METHODS OF STRUCTURAL MECHANICS. FINITE ELEMENT METHOD

15.1. Potential Field of Force. Potential Eenergy

A field of force is a space in which a certain force acts on a material point placed there.

This concept is a general one. Examples of force fields are the gravitational fields of planets, the magnetic field of an object, an electrostatic field, etc. A special place among them is taken by potential force fields that have two important physical properties: 1) the force of this field is positional force, that is $F = F(x, y, z)$; 2) the work of the field force does not depend on the trajectory along which the force applied to a certain point moves, but depends only on the positions of the start and end points; it can be calculated through the integral sum of the corresponding elementary works:

$$A_{(M_1 M_2)} = \int_{(M_1)}^{(M_2)} \left(F_x\, dx + F_y\, dy + F_z\, dz\right) . \qquad (15.1)$$

Forces acting in a potential force field are called potential.

If the expression (15.1) under the sign of the integral is the full differential of some function $U(x, y, z)$, that is:

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz = F_x dx + F_y dy + F_z dz , \qquad (15.2)$$

then the function U is called the force-function.

Taking into account the last condition, we obtain:

$$A_{(M_1 M_2)} = \int_{(M_1)}^{(M_2)} dU(x, y, z) = U_2 - U_1 . \qquad (15.3)$$

The work of the potential force is equal to the difference in the values of the force function at the final and initial points of the path of force motion. From relation (15.2) it follows that the force function is found from the equality:

$$U = \int (F_x\, dx + F_y\, dy + F_z\, dz) + C\,.$$

The constant C can have any value. As can be seen from equality (15.3), the work of force does not depend on C.

Gravity forces, as well as the elastic forces of an elastic body, are both potential in an adiabatic process (i.e., a process that takes place without heat exchange with the environment) and isothermal processes (i.e., processes that occur in a physical system at a constant temperature). For these forces there are force-functions.

So, for gravity force F, directed along the axis z (the axis z is directed vertically upwards), we have $F_z = -F$ and $dA = -F\, dz$. Taking $U = 0$ when $z = 0$ we receive:

$$U = -F\, z\,.$$

The elastic force in the centrally stretched rod (Figure 15.1) is directed in the direction opposite to the external force F.

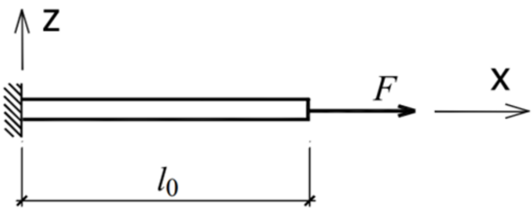


Figure 15.1

Therefore $F_x = -F = -r\,x$ (r – is the rigidity coefficient of the elastic rod). The elementary work of this force is equal to $dA = -r\,x\,dx$. Counting $U = 0$ at $x = 0$, we find:

$$U = -\frac{1}{2}\, r\, x^2\,.$$

In a potential force field, the projections of forces are equal to the partial derivatives of the force function relative to the corresponding co-ordinates. Indeed, it follows from equality (15.2) that:

$$F_x = \frac{dU}{dx}, \quad F_y = \frac{dU}{dy}, \quad F_z = \frac{dU}{dz}.$$

Determining the mixed derivatives for U, we find:

$$\frac{dF_x}{dy} = \frac{d^2U}{dxdy}, \quad \frac{dF_y}{dx} = \frac{d^2U}{dxdy}, \text{ etc.}$$

Consequently,

$$\frac{dF_x}{dy} = \frac{dF_y}{dx}, \quad \frac{dF_y}{dz} = \frac{dF_z}{dy}, \quad \frac{dF_z}{dx} = \frac{dF_x}{dz}.$$

These relations are necessary and sufficient conditions for the potentiality of the force field.

The potential energy at the given point M in the field is the amount of work that the force field would have done when moving a material point from a given position to one in which the potential energy is conventionally assumed to be zero (point O):

$$\Pi = A_{(MO)}.$$

Since the functions $\Pi(x, y, z)$ and $U(x, y, z)$ have the same null values (follows from the definitions), from (15.3) when $U_0 = 0$ we obtain:

$$A_{(MO)} = U_0 - U = -U,$$

where U is the value of the force-function at the point M.

Thus, we obtain:

$$\Pi(x, y, z) = -U(x, y, z).$$

The work of potential force can be calculated not by expression (15.3), but by the formula:

$$A_{(M_1 M_2)} = \Pi_1 - \Pi_2 ,$$

that is, it is equal to the difference in the values of potential energy in the initial and final positions of the point.

Work and energy, of course, are measured in the same units. Remember that in the SI system the basic units are: meter (m) – unit of length, kilogram (kg) – unit of mass, second (s) – unit of time. The unit of work and energy is the joule (J). 1 J is equal to work that is accomplished by force of 1 N in the path of 1 m.

The technique often uses the MKGFS system. The unit of work is 1 kilogram–force–meter (1 kgf·m). It is the work that is performed with a force of 1 kgf on a distance of 1 m.

Relations between units: 1 kGf·m = 9.81 J; 1 J = 0.102 kGf·m.

15.2. Potential Energy of Elastic System Deformation

A special case of the general definition of potential energy given in Section 15.1 is the determination of the potential energy of an elastic deformed body, that is, the field of elasticity forces.

The potential deformation energy U of an elastic system is the amount of work that internal forces would have done in transferring the system from a deformed state to an undeformed one; it is the energy of elasticity forces. It is equal in absolute value, but opposite in sign to the actual work of internal forces, i.e.:

$$U = -A_{\text{int}} .$$

In particular, for a linearly elastic rod under tension-compression:

$$U^{(N)} = \frac{1}{2}\int_0^l \frac{N^2 dx}{EA} .$$

And in pure bending:

$$U^{(M)} = \frac{1}{2}\int_0^l \frac{M^2 dx}{EJ} .$$

In general, for a plane bars system:

$$U = \frac{1}{2}\sum\int\frac{M^2 dx}{EJ} + \frac{1}{2}\sum\int\frac{N^2 dx}{EA} + \frac{1}{2}\sum\int\frac{\mu Q^2 dx}{GA}.$$

In these expressions, U is written through efforts (internal forces).

One can represent U through functions expressing the displacements of the points (cross-sections) of the bars. For example, using differential dependencies:

$$N = EA\,u' \text{ and } M = EJ\;y'',$$

we get:

$$U^{(N)} = \frac{1}{2}\int_0^l EA\,u'^2 dx,$$

$$U^{(M)} = \frac{1}{2}\int_0^l EJ\;y''^2 dx.$$

In some cases, the deformation energy of the bar is conveniently expressed not through the functions $u(x)$ or $y(x)$, but through the displacements of individual cross-sections.

For the tensioned bar (Figure 15.2), the horizontal movement of the end of the bar is determined by the parameter Δl. Then, calculating the potential energy through the work of external forces, we obtain:

$$U^{(N)} = \frac{1}{2}F\,\Delta l = \frac{1}{2}\left(\frac{EA}{l}\,\Delta l\right)\Delta l = \frac{1}{2}\frac{(\Delta l)^2}{l}\,EA.$$

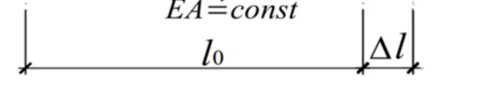


Figure 15.2

For a bended bar (Figure 15.3), the force F, causing the displacement Δ, is equal to

$$F = \frac{3EJ}{l^3}\Delta .$$

Consequently,

$$U^{(M)} = \frac{1}{2}F\Delta = \frac{1}{2}\left(\frac{3EJ}{l^3}\Delta\right)\Delta = \frac{3}{2}\frac{\Delta^2}{l^3}EJ .$$

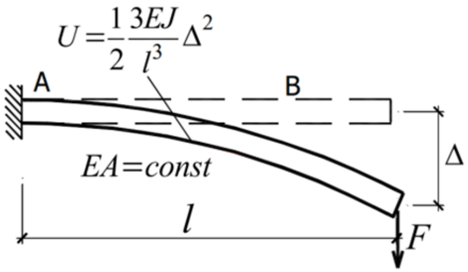


Figure 15.3

So, the energy of elastic deformation can be expressed through efforts, through the functions of displacements, or through discrete parameters of displacements.

Note: In some questions of mechanics, the concept of specific potential energy U_0 (in other words, energy density) is used. It is equal to the area bounded by the $\sigma - \varepsilon$ curve, the ε axis and the vertical corresponding to the final value of the relative deformation (Figure 15.4).

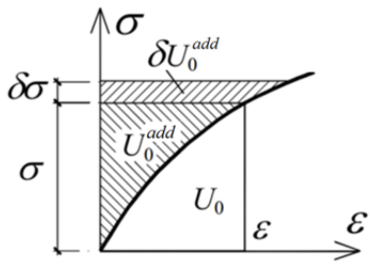


Figure 15.4

The potential deformation energy of the body is calculated through the specific energy by the expression:

$$U = \iiint_V U_0 \, dx \, dy \, dz \, .$$

By U_0^{add} in this figure, additional potential energy (additional work) is denoted. For a linearly elastic bar

$$U = U^{add} \, .$$

15.3. Generalized Displacements and Forces. Derivatives of Potential Energy Expressions

The deformation energy U, equal to the work of external forces, is determined by the equality:

$$U = \frac{1}{2}\left(F_1 \, \Delta_1 + F_2 \, \Delta_2 + \ldots + F_n \, \Delta_n\right) = \\ = \frac{1}{2}\left[F_1 \;\; F_2 \ldots F_n\right] \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} = \frac{1}{2} \vec{F}^T \, \vec{\Delta}. \tag{15.4}$$

Since $\vec{\Delta} = A \cdot \vec{F}$, then:

$$U = \frac{1}{2} \vec{F}^T A \, \vec{F} \, . \tag{15.5}$$

It is obteined matrix representation of the quadratic form of n variables $F_1, \, F_2, \ldots, F_n$, where the matrix of the quadratic form is denoted with A:

$$A = \begin{bmatrix} \delta_{11} & \delta_{12} & \ldots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \ldots & \delta_{2n} \\ \vdots & & & \\ \delta_{n1} & \delta_{n2} & \ldots & \delta_{nn} \end{bmatrix}.$$

If in the formula (15.5) the result of the multiplication is represented in the scalar form of the record, then we obtain:

$$U=\frac{1}{2}\Big(\delta_{11}\,F_1^2+\delta_{12}\,F_1\,F_2+\delta_{13}\,F_1\,F_3+\ldots+\delta_{1n}\,F_1\,F_n+$$
$$\delta_{21}\,F_2\,F_1+\delta_{22}\,F_2^2+\ldots+\delta_{2n}\,F_2\,F_n+\ldots \tag{15.6}$$
$$+\delta_{n1}\,F_n\,F_1+\ldots+\delta_{nn}\,F_n^2\Big)=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ij}F_iF_j.$$

The potential energy of the system is always positive. Therefore, the recorded quadratic form cannot become negative at any value $F_1,\ F_2,\ldots,F_n$. Such quadratic forms are called positive definite.

Expression (15.4) can be represented as:

$$U=\frac{1}{2}\,\vec{\Delta}^T\vec{F}\,. \tag{15.7}$$

The vector $\vec{F}$ can be expressed through the external rigidity matrix:

$$\vec{F}=R\,\vec{\Delta}\,,\ R=A^{-1}\,.$$

With this in mind, the deformation energy can be written as:

$$U=\frac{1}{2}\,\vec{\Delta}^T\ R\ \vec{\Delta}\,. \tag{15.8}$$

A matrix record of a quadratic form is obtained through generalized displacements. In the formula (15.8), the matrix of the external rigidity of the system

$$R=\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & & & \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}$$

is a quadratic matrix.

In the scalar form of the record, we obtain:

$$U = \frac{1}{2}\Big(r_{11}\,\Delta_1^2 + r_{12}\,\Delta_1\,\Delta_2 + \ldots + r_{1n}\,\Delta_1\,\Delta_n + \\ + r_{21}\,\Delta_2\,\Delta_1 + r_{22}\,\Delta_2^2 + \ldots + r_{nn}^2\,\Delta_n^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}\Delta_i\Delta_j . \qquad (15.9)$$

We differentiate expression (15.6) relative to the variable F_1. Given the property of displacements reciprocity $\delta_{ik} = \delta_{ki}$ (matrix A is symmetric), we obtain:

$$\frac{\partial U}{\partial F_1} = \delta_{11}\,F_1 + \delta_{12}\,F_2 + \delta_{13}\,F_3 + \ldots + \delta_{1n}\,F_n = \Delta_1 .$$

In general:

$$\frac{\partial U}{\partial F_i} = \Delta_i . \qquad (15.10)$$

This expression is a record of Castiliano's first theorem (1875): **the derivative of the potential strain energy relative to force is equal to the displacement of the point of application of this force in its direction**.

Differentiating expression (15.9) relative to the variable Δ_1 and taking into account equality $r_{ik} = r_{ki}$ (matrix R is symmetric), we obtain:

$$\frac{\partial U}{\partial \Delta_1} = r_{11}\,\Delta_1 + r_{12}\,\Delta_2 + \ldots + r_{1n}\,\Delta_n = F_1 .$$

In general:

$$\frac{\partial U}{\partial \Delta_i} = F_i . \qquad (15.11)$$

This expression is a record of the second Lagrange theorem: **in the equilibrium position, the derivative of the potential strain energy relative to displacement is equal to the corresponding force.**

15.4. Total Energy of the Deformable System

From the energy view point, the phenomenon of the body deformation is a process of energies exchange of two systems of forces (force fields): internal and external.

Therefore, for a complete energy characteristic of a body in a deformed state, it is not enough to consider only the deformation energy U, since it represents a part of the energy of the interacting force fields.

We will consider only conservative external forces. Their work depends only on the initial and final state and does not depend on the way of transition from one position to another. Conservative forces include, for example, gravity forces.

If we take the energy of the system in the initial (undeformed) state equal to zero, then the potential Π of external forces in the deformed state will be measured by the amount of work that these forces can perform when the system is transferred from given state to the initial one.

The total energy of the loaded body is taken equal to:

$$E = U + P, \tag{15.12}$$

where U is the potential energy of deformation (elastic potential or, otherwise, the energy of elastic forces, the potential of internal forces);

P is the energy of external forces (potential of external forces).

External forces are gravity forces. With a relatively small change in the distance between bodies in near-Earth space, gravitational forces practically do not change. Therefore, gravity forces form a homogeneous force field, that is, a field in which the value of each force is constant, independent of the displacements of their application points. Their work is calculated as the work of unchanging forces when moving the system from a given position to the initial one.

For a centrally tensioned rod (Figure 15.5)

$$P = -F\,\Delta l\,,$$

and for a bended bar loaded with a distributed load (Figure 15.6):

$$P = -\int_0^l q(x)\, y(x)\, dx \, .$$

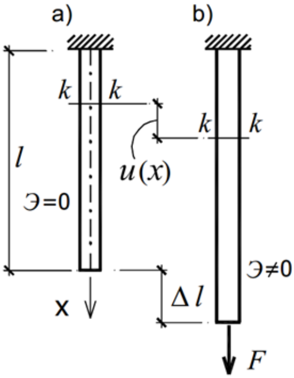


Figure 15.5

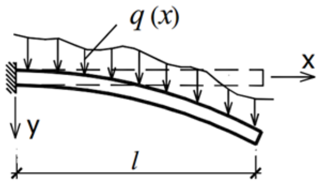


Figure 15.6

Thus, the total energy of the system can be expressed either through the functions of displacements or through discrete parameters.

For the last example:

$$E = U + P = \frac{1}{2}\int_0^l EJy''^2\, dx - \int_0^l q(x)\, y\, dx \, .$$

As you can see, the value E depends on the function $y(x)$, that is, it is a functional (function of function) $E = E(y)$.

For a discrete linear-elastic system, the potential of internal forces is (see formula (15.9)):

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}\Delta_i\Delta_j \, .$$

Replacing the notation of the generalized displacement Δ by Z, we obtain:

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}Z_iZ_j .$$

The potential of external forces is:

$$P = -\sum_{i=1}^{n} F_i \, Z_i = \sum_{i=1}^{n} R_{iF} \, Z_i \,, \text{ since } \; R_{iF} = -F_i \,,$$

where R_{iF} are reactions in additional constraints of the primary system of the displacement method.

Then the expression for the total energy of the system is represented as:

$$E = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} \, Z_i \, Z_j + \sum_{i=1}^{n} R_{iF} \, Z_i \quad . \tag{15.13}$$

15.5. Displacements Variation Principle

This principle expresses the equilibrium condition of a deformable system, recorded through its displacements, using the introduced concept of total energy E.

For a tensioned rod (Figure 15.5) $u(x)$ is a function that determines the longitudinal displacements of the cross-sections; $u(x)$ are true displacements for which the balance between external and internal forces is established.

In a deformed state, the total energy of the rod is equal to the work of internal and external forces on displacements $(-u)$:

$$E(u) = U + P = -(A_{\text{int}} + W_{ext})$$

Let us give the points of the system additional infinitesimal displacements $\delta u = \delta u(x)$; δu is an arbitrary function with infinitesimal ordinates. It is called a variation of function $u(x)$.

In a state $u + \delta u$, the energy will be equal to:

$$E(u + \delta u) = -(A_{\text{int}} + \delta A_{\text{int}} + W_{ext} + \delta W_{ext}) \,.$$

Subtracting the expression $E(u)$ from the last equality, we obtain an infinitesimal change in energy (the first variation of energy) caused by the variation of the function δu:

$$\delta E = E(u+\delta u) - E(u) = -(\delta A_{\text{int}} + \delta W_{ext}) .$$

For a system that is in equilibrium, when the displacements $u(x)$ occurs, the right-hand side in the last equality is equal to zero, since, in accordance with the principle of virtual displacements (see Section 7.4), the work of all the forces of the system on virtual displacements δu must be equal to zero:

$$\delta A = \delta A_{\text{int}} + \delta W_{ext} = 0 ,$$

therefore,

$$\delta E = 0 . \tag{15.14}$$

This is a formal notation of the displacements variation principle (the Lagrange principle): of all the displacements allowed by the constraints of the system, the true displacements $u(x)$ **have the property that the total energy of the system has a stationary value when these displacements occure.** Such a property of energy will be observed when it has an extreme value for the true displacements in comparison with all nearest ones.

Consider the scheme shown in Figure 15.7.

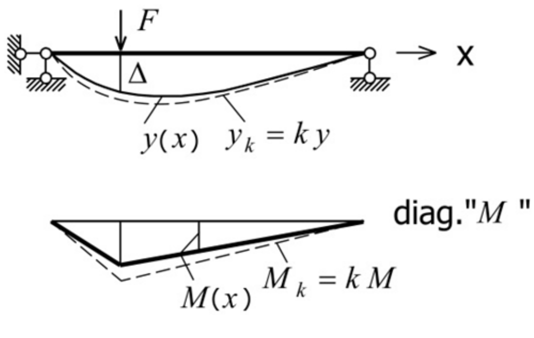


Figure 15.7

$$E = U(M) + P = \frac{1}{2}\sum\int\frac{M^2 dx}{EJ} - F\ \Delta = \frac{1}{2}F\Delta - F\Delta$$

or

$$E = U(y) + P = \frac{1}{2}\sum\int EJ\ y''^2 dx - F\ \Delta = \frac{1}{2}F\Delta - F\Delta\ .$$

The terms in these expressions are converted on the basis of the numerical equality of the potential energy of elastic deformation and the actual work of external forces.

We investigate the change in the total energy of the system depending on the change (variation) of the deformed beam axis. For example, we increase the ordinates of the deflections of the beam axis by a factor of *k*. We get:

$$E = \frac{k^2}{2}\sum\int EJ\ y''^2 dx - kF\ \Delta = \frac{k^2}{2}F\Delta - kF\Delta = F\Delta\left(\frac{k^2}{2} - k\right).$$

Energy is represented by a function of the second degree of k. A graphic illustration of the dependence $\frac{E}{F\ \Delta}(k)$ is shown in Figure 15.8.

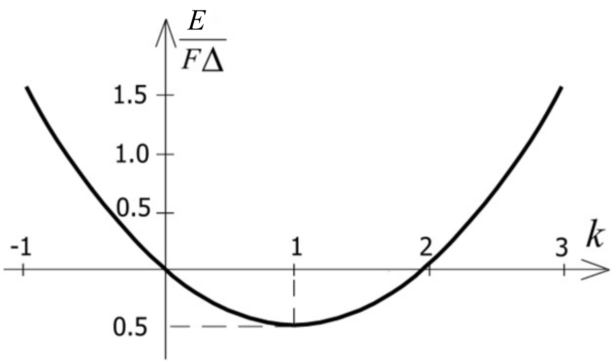


Figure 15.8

When $k = 1$, that is, in the real state of equilibrium, $\min E(y)$ takes place.

In this example, it would be possible to vary not the equation of the bent beam axis, but the corresponding function of bending moments, i.e., the stress state.

The result of the calculations would naturally be the same.

Let us consider a second example. In a discrete linearly deformable system under one-parameter loading, all generalized parameters are interconnected linearly. Therefore, using the parameter of a generalized displacement Z, we can write the total energy in the form:

$$E = \frac{1}{2} Z^2 \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} \, \overline{Z}_i \, \overline{Z}_j + Z \sum_{i=1}^{n} R_{iF} \, \overline{Z}_i ,$$

where $\overline{Z}_i, \overline{Z}_j$ are the components of the basis vector of the system displacements, corresponding to the unit parameter of the generalized load F.

Since there is equality for the system:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} \, \overline{Z}_i \, \overline{Z}_j = - \sum_{i=1}^{n} R_{iF} \, \overline{Z}_i ,$$

then the expression for energy can be represented in this form of notation:

$$\frac{E}{\Lambda} = \frac{Z^2}{2} - Z ,$$

where

$$\Lambda = \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} \, \overline{Z}_i \, \overline{Z}_j .$$

Function $\frac{E}{\Lambda}(Z)$ has a minimum at a point (1.0; –0.5).

We increase the displacement Z by a factor k. Then, given that for the final load value the parameters F and Z are both fixed, expressing through the actual work of external forces, we obtain:

$$E = \frac{k^2}{2} Z^2 \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij}\, \overline{Z}_i\, \overline{Z}_j + k\, Z \sum_{i=1}^{n} R_{iF}\, \overline{Z}_i \quad =$$

$$= k^2\, \frac{1}{2}\, F\, Z - k\, F\, Z = F\, Z \left(\frac{k^2}{2} - k \right).$$

The dependence $\frac{E}{FZ}(k)$ has the same form as for the beam. This leads to the conclusion that of all possible deformed states of the system, the true state occurs at $k = 1$. The total energy of the system in this state is minimal.

Investigation of the behavior of functionals E at stationary points using the second variation $\delta^2 E$ gives reason to judge the quality of the system equilibrium. P.G.L. Dirichlet (German mathematician, 1805–1859) proved that:

– if $\delta E = 0$ and $\delta^2 E > 0$, then $E = E_{\min}$ (stable equilibrium);

– if $\delta E = 0$ and $\delta^2 E > 0$, then $E = E_{\max}$ (unstable equilibrium);

– if $\delta E = 0$ and $\delta^2 E > 0$, then $E = \text{const}$ (indifferent equilibrium).

A thorough study of the equilibrium states of mechanical systems will be carried out in the section "Stability of structures".

In the problems of the structural statics, methods for calculating stable systems are studied. Therefore, the stationarity condition $\delta E = 0$ for them is identified with the condition of minimum total energy.

15.6. Ways to Solve Variational Problems

The functions $y(x)$, that realize the extremum of the functional $E(y)$, can be found in two ways:

1. By solving differential equations obtained from condition $\delta E = 0$ (15.14).
2. Using the so-called direct methods of variations calculus.

The problem of finding $y(x)$ by solving the differential equation is addressed in those cases when, for the element (object) under study, the energy can be written as a function depending on the displacements and their derivatives of the first, second, and higher order. A necessary condition for the minimum of a definite integral

$$E = \int_a^b \Phi\left(y, y', y'', \ldots, y^{(\kappa)}\right) dx ,$$

that is, the stationarity condition

$$\delta E = \int_a^b \delta\Phi \, dx = 0$$

is reduced for an arbitrary choice of function δy to the Euler – Lagrange differential equation:

$$\frac{\partial\Phi}{\partial y} - \frac{d}{dx}\left(\frac{\partial\Phi}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial\Phi}{\partial y''}\right) + \ldots + (-1)^{\kappa} \frac{d^{\kappa}}{dx^{\kappa}}\left(\frac{\partial\Phi}{\partial y^{(\kappa)}}\right) = 0 .$$

As an example, we show the solution to the problem of bending the cantilever beam on an elastic Winkler base (Figure 15.9).

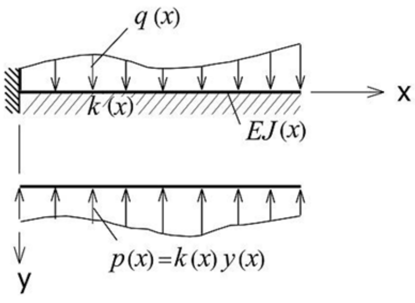


Figure 15.9

We determine the total energy of the interacting forces:

$$E = \int_0^l \left(\frac{EJ}{2} y''^2 + \frac{k\, y^2}{2} - q\, y \right) dx = \int_0^l \Phi\left(y, y''\right) dx .$$

In this case:

$$\frac{\partial \Phi}{\partial y} = k\ y - q\,, \qquad \frac{\partial \Phi}{\partial y'} = 0\,, \qquad \frac{\partial \Phi}{\partial y''} = EJ\ y''\,.$$

The differential equation corresponding to condition $\delta E = 0$, will have the form:

$$\left(EJ\ y''\right)'' + k\ y = q\,.$$

For $k = 0$ we get the usual differential equation of transverse bending:

$$\left(EJ\ y''\right)'' = q\,.$$

The general solution of the equation will contain four arbitrary constants. To obtain a particular solution, four additional conditions must be set.

Direct methods of variations calculus allow us to reduce the problem of finding the functional minimum to the problem of finding the minimum of a function of many variables by solving a system of linear algebraic equations. These include the Rayleigh – Ritz, Bubnov – Galerkin methods, the callocation method, etc. Let us show the essence of direct methods using the example of the Rayleigh – Ritz method.

From an infinite system of functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_r(x), \ldots$, satisfying the boundary conditions of the problem, we select the first r functions $\varphi_i(x)$ and form a new function f_r of the following form, using a linear combination:

$$f_r(x) = a_1\ \varphi_1(x) + a_2\ \varphi_2(x) + \ldots + a_r\ \varphi_r(x) = \sum_{i=1}^{r} a_i\ \varphi_i(x)\,,$$

where a_i are the arbitrary coefficients. Functions $\varphi_i(x)$ are called coordinate or basic functions.

The functional $E(\Phi(x))$ after replacing $\Phi(x)$ by $f_r(x)$ turns into a function $E(a_1, a_2, \ldots, a_r)$ of r independent variables. The necessary condition for the extremum of a function of several variables is the zeroing of partial derivatives of the first order, i.e.:

$$\frac{\partial E}{\partial a_i} = 0 \qquad (i = 1, 2, ..., r). \tag{15.15}$$

Solving the system of equations (15.15), we find the values of the parameters a_i, and hence obtain an approximate solution of the stationarity conditions $\delta E = 0$.

15.7. Calculating Elastic Systems Based on the Displacement Variation Principle

The displacement of any point (cross-section) of an element (Figure 15.10), taking into account generally accepted assumptions, can be unambiguously expressed through nodal (generalized) displacements. Thus, the horizontal displacement of cross-section *C*, as follows from Figure 15.11 can be determined by the formula:

$$u = Z_1 \left(1 - \frac{x}{l}\right) + Z_4 \frac{x}{l} = Z_1 f_1(x) + Z_4 f_4(x), \tag{15.16}$$

where $f_1(x)$, $f_4(x)$ are the basis (coordinate) functions.

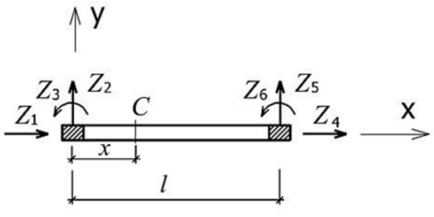


Figure 15.10

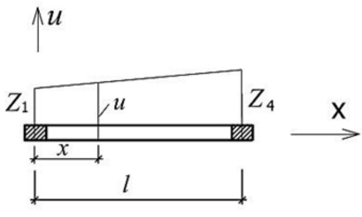


Figure 15.11

To determine the displacements caused only by nodal displacements Z_2, Z_3, Z_5 and Z_6, we use the differential equation:

$$\frac{d^4 v}{dx^4} = 0 .$$

Its general solution has the form:

$$v = C_1 x^3 + C_2 x^2 + C_3 x + C_4 .$$

We find, for example, the equation of the bended axis caused by loading the bar in the form $Z_2 = 1$. The boundary conditions for this case:

$$x = 0 \quad v = 1; \quad x = 0 \quad v' = 0 ;$$

$$x = l \quad v = 0; \quad x = l \quad v' = 0 .$$

Having solved the fourth-order system of equations, we obtain the values of arbitrary constants C_1, C_2, C_3, C_4. The equation of the bended axis is written as:

$$v = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} .$$

For other unit nodal displacements of the bar clamped at the ends, the deflection curves are recorded in Table. 15.1.

Table 15.1

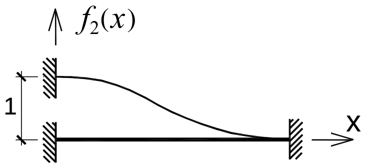
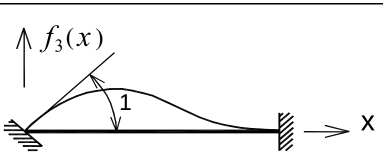
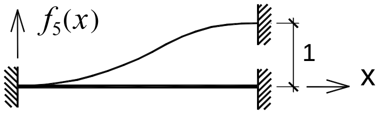
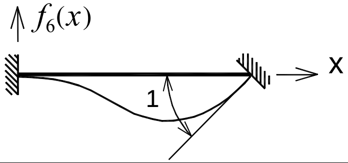
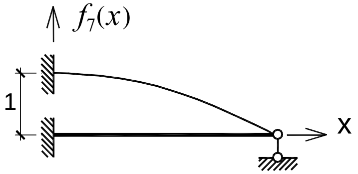
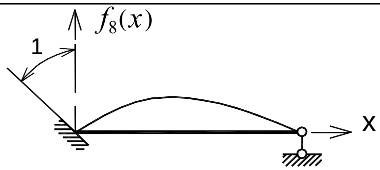
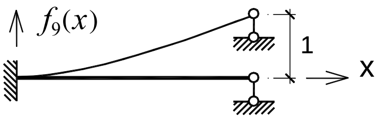
№	The scheme of the bar. Kind of displacement	Bended axis equation
1	2	3
1		$f_2(x) = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}$
2		$f_3(x) = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$

Table 15.1 (ending)

1	2	3
3		$f_5(x)=\frac{3x^2}{l^2}-\frac{2x^3}{l^3}$
4		$f_6(x)=-\frac{x^2}{l}+\frac{x^3}{l^2}$
5		$f_7(x)=1-\frac{3x^2}{2l^2}+\frac{x^3}{2l^3}$
6		$f_8(x)=x-\frac{3x^2}{2l}+\frac{x^3}{2l^2}$
7		$f_9(x)=\frac{3x^2}{2l^2}-\frac{x^3}{2l^3}$

Using the forces action independence principle, the vertical displacement of the cross-section C (Figure 15.9) can be represented as:

$$v=Z_2\left(1-\frac{3x^2}{l^2}+\frac{2x^3}{l^3}\right)+Z_3\left(x-\frac{2x^2}{l}+\frac{x^3}{l^2}\right)+$$

$$+Z_5\left(\frac{3x^2}{l^2}-\frac{2x^3}{l^3}\right)+Z_6\left(-\frac{x^2}{l}+\frac{x^3}{l^2}\right)= \qquad (15.17)$$

$$=Z_2\ f_2(x)+Z_3\ f_3(x)+Z_5\ f_5(x)+Z_6\ f_6(x).$$

The expression for determining the rotation angle of the cross-section is obtained by differentiating $v = v(x)$ relative to x.

For a bar clamped at one end and hinge-supported at the other end, the deflection curves for unit nodal displacements and the corresponding displacements functions are also shown in Table. 15.1.

In the general case, for a discrete system, the expression for determining the displacement of a certain point can be represented as:

$$Z = Z_1 \, f_1(s) + Z_2 \, f_2(s) + ... + Z_n \, f_n(s) \,, \qquad (15.18)$$

where $f_i\left(s\right)$ are the basic functions corresponding to the generalized displacements Z_i.

The number of such equations corresponds to the number of deformable elements of the system and the number of displacements types (linear, angular).

For bars systems, these equations will be accurate, for continuum systems they will be approximate.

In connection with the above, the total energy of the system can be represented as a function of *n* generalized displacements (coordinates) and load:

$$E = E(Z_1, Z_2, ..., Z_n, F) \,.$$

Then the stationarity condition:

$$\delta E = \frac{\partial E}{\partial Z_1}\delta Z_1 + \frac{\partial E}{\partial Z_2}\delta Z_2 + ... + \frac{\partial E}{\partial Z_n}\delta Z_n = 0$$

with independent variations δZ_i and constant load F will allow us to get n equations to determine Z_i:

$$\begin{aligned} &\frac{\partial E}{\partial Z_1} = \frac{\partial U}{\partial Z_1} + \frac{\partial P}{\partial Z_1} = 0, \\ &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ &\frac{\partial E}{\partial Z_n} = \frac{\partial U}{\partial Z_n} + \frac{\partial P}{\partial Z_n} = 0. \end{aligned} \qquad (15.19)$$

For a linearly elastic system, the total energy is calculated by the formula (15.13), therefore, equations (15.19) in the expanded form of writing take the form of canonical equations of the displacement method:

$$r_{11}Z_1 + r_{12}Z_2 + \ldots + r_{1n}Z_n + R_{1F} = 0,$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$r_{n1}Z_1 + r_{n2}Z_2 + \ldots + r_{nn}Z_n + R_{nF} = 0.$$

For nonlinearly deformable systems, equations (15.19) will be nonlinear relative to Z_i.

Due to the characters of equations (15.18) for continuum systems, only approximate values Z_i may be determined. In this case, equations (15.19) are called the equations of the Ritz method, which, as it was noted earlier, refers to direct methods of variations calculus.

15.8. The Essence of the Finite Element Method

The finite element method (FEM) is an effective numerical method for solving applied problems and is widely used to analyze various structures. This method is well adapted to computer implementation. According to a single methodology, bars, plates, massives and combined structural systems are calculated. Its essence is as follows. The system under study is mentally divided into many finite elements (disjoint areas), that is, a transition is made from a given design scheme to a discrete one.

The shape of the finite element (FE) is predetermined by the features of the analysed object (system, structure). In bars systems, FE is taken as a bar (as a rule) with constant longitudinal and bending rigidity. For plates and thin-walled spatial continuum systems, triangular or rectangular (ge-nerally quadrangular) finite elements are most often used; for solving three-dimensional problems, volume finite elements in the form of a tetrahedron or parallelepiped are used. The choice of the shape and sizes of finite elements has a significant effect on the calculation results. But the results, of course, should allow you correctly evaluate the stress-strain state of the initial system. Representation of the system under study as a sufficiently large set of finite elements leads to an increase in the calculation accuracy, but significantly increases the dimension of the problem, what is associated with a significant amount of calculations.

Readers who are interested in questions of estimation of discretization error can find relevant recommendations in the scientific literature.

The points at which FEs are connected are called nodes. We need to distinguish rigid and hinge nodes. In a rigid node, it is assumed that there are constraints ensuring the continuity of linear and angular FE displacements adjacent to this node. The constraints in the hinge node allow saving the continuity of linear movements. Nodal displacements and the corresponding nodal forces are taken as generalized.

The idea of the method is to describe the stress-strain state of the FE through generalized displacements Z of nodes and to establish their connection with the load acting on the system. To implement this idea, it is necessary to obtain a FE stiffness matrix.

Since the displacements function of the original system is unknown, it must be set. If in the Ritz method it was assumed that the basis functions are determined by one expression on the entire region of the system, then an alternative approach is implemented in the FEM. It consists in the fact that at each FE unknown functions of displacements are replaced by approximating ones so that the displacements of all element points are expressed through the nodal ones. For one-dimensional elements, taking into account the remark about the rigidity constancy, the displacement function is accurate (see Section 15.7), for two-dimensional and three-dimensional FEs, these functions are written approximately, most often in the form of polynomials. Their selection is a rather difficult task. The accuracy of the final results substantially depends on the successful solution of this problem. Using approximating functions and based on the variational principles of structural mechanics, one of the main tasks of the FEM is solved – the determination of stiffness matrices of finite elements.

Since each bar in the composition of the system under study has its own orientation, stiffness matrices are first constructed in the local coordinate system, and then, when moving from the local system to the general, they are transformed. The stiffness matrix of the entire system is obtained by the corresponding combination of stiffness matrices of individual elements.

The resolving equations of the FEM are written in the form:

$$R\,\vec{Z} + \vec{R}_F = 0\,, \tag{15.20}$$

where $\vec{R}_F$ is the vector of reactions caused by the given load, which is equal to the vector of nodal loads taken with the opposite sign.

The total load on the node is defined as the sum of the loads from the elements adjacent to the node. Since the non-nodal load is replaced by the equivalent nodal load in the directions $\vec{Z}$, then the vector of rections

$$\vec{R}_F = \left[R_{1F}, R_{2F}, \ldots, R_{nF}\right]^T = -\vec{F},$$

where

$$\vec{F} = \left[F_1, F_2, \ldots, F_n\right]^T$$

is the vector of nodal loads.

After solving the system of equations (15.20), the displacements Z of nodes in the general coordinate system become known. To calculate the forces in a finite element, it is convenient to first find the nodes displacement vector $\vec{Z}'$ in the local coordinate system, and then determine the reactions at the ends of the FE using the stiffness matrix R'. For-mulas for the corresponding transformations are given in sections 15.9 and 15.10.

This form of calculation corresponds to the FEM variant “in displacements”. It is the most common form.

Another approach to solving the problem with the help of FEM is also possible. The stress-strain state of the FE must be described by a finite set of generalized nodal forces, and then establish their relationship with the load. This form of calculation corresponds to the FEM “in efforts”.

15.9. Bar Stiffness Matrix in the Local Coordinate System

There are several ways to obtain stiffness matrices of separate bars. One of the simplest is a method based on the known conditions of the displacement method.

Each end of the bar adjacent to the rigid node has three degrees of freedom: linear displacements in the horizontal and vertical directions and the angle of rotation. The force factors corresponding to these displacements are the reactive forces R_1', R_2', R_4', R_5' and the moments R_3', R_6', located at the FE edges. (Displacements, reactions, stiffness matrix and its elements in the local coordinate system are indicated by letters with strokes). The stiffness matrix (the matrix of unit reactions) converts the displacement vector

$$\vec{Z}' = \left[Z'_1, Z'_2, Z'_3,\ Z'_4, Z'_5,\ Z'_6\right]^T$$

into the vector of reactions at the ends of FE, i.e., there is the relation:

$$\begin{bmatrix} R'_1 \\ R'_2 \\ \vdots \\ R'_6 \end{bmatrix} = \begin{bmatrix} r'_{11} & r'_{12} & \cdots & r'_{16} \\ r'_{21} & r'_{22} & \cdots & r'_{26} \\ . & . & . & . \\ r'_{61} & r'_{62} & \cdots & r'_{66} \end{bmatrix} \begin{bmatrix} Z'_1 \\ Z'_2 \\ \vdots \\ Z'_6 \end{bmatrix} = R'\,\vec{Z}'\,.$$

The positive directions of the reactions R'_i correspond to the positive directions of Z'_i .

Elements of the matrix R' are reactions in constraints caused by unit displacements $Z'_i = 1$ (Figure 15.12).

In the first column, the reaction values caused by $Z'_1 = 1$ are recorded, in the second column – caused by $Z'_2 = 1$ etc. Therefore, to calculate the elements of R' matrix, we can use the data from the table used in the displacement method to determine the reactions in the supports of a bar with constant cross-section.

The matrix R' has the form:

$$R' = \begin{bmatrix} \frac{EA}{l} & & & -\frac{EA}{l} & & \\ & \frac{12EJ}{l^3} & \frac{6EJ}{l^2} & & -\frac{12EJ}{l^3} & \frac{6EJ}{l^2} \\ & \frac{6EJ}{l^2} & \frac{4EJ}{l} & & -\frac{6EJ}{l^2} & \frac{2EJ}{l} \\ -\frac{EA}{l} & & & \frac{EA}{l} & & \\ & -\frac{12EJ}{l^3} & -\frac{6EJ}{l^2} & & \frac{12EJ}{l^3} & -\frac{6EJ}{l^2} \\ & \frac{6EJ}{l^2} & \frac{2EJ}{l} & & -\frac{6EJ}{l^2} & \frac{4EJ}{l} \end{bmatrix}. \quad (15.21)$$

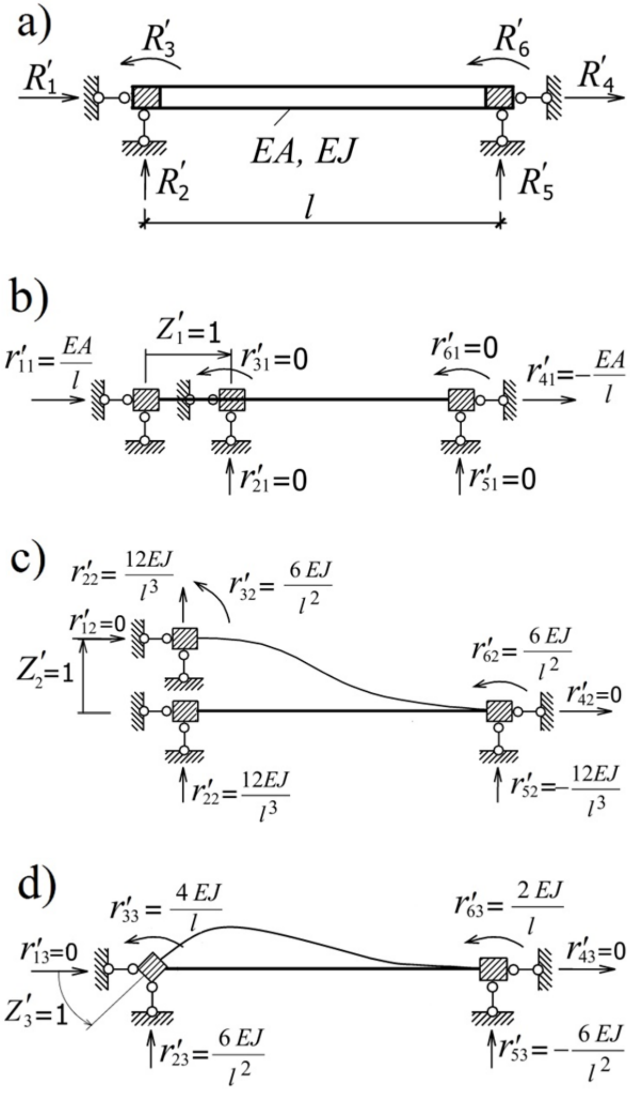


Figure 15.12

For bars with other fixing conditions, matrix R' elements are calculated similarly.

Consider methods based on the use of approximating displacement functions. The necessary calculation procedure for one of them, for example, for a bar fixed at the ends is as follows.

1. In a linearly deformable bar the longitudinal and transverse displacements of the cross-sections are not interconnected. Therefore, the functions that describe the character of the change in displacements along the length of the bar will be different for them. In accordance with the differential equation

$$N = E\,A\,u',$$

we will approximate the displacements of the bar cross-sections along its axis by a linear function:

$$u(x) = a_1 + a_4\,x\,. \tag{15.22}$$

The bent axis of the bar in the absence of distributed load along its length is described by a third-order curve, which is a consequence of the differential equation

$$\nu^{(IV)} = 0\,.$$

Therefore, the approximating polynomial of the third degree allows us to precisely set the function of the displacements.

Let us assume that

$$v(x) = a_2 + a_3 x + a_5 x^2 + a_6 x^3\,. \tag{15.23}$$

There are unknown parameters a_i in expressions (15.22) and (15.23); their number is equal to the number of degrees of freedom.

Functions $u(x)$ and $\nu(x)$ are called form functions

The rotation angle of the bar cross-section is determined by the value of the first derivative:

$$\frac{d\nu}{dx} = a_3 + 2a_5 x + 3a_6 x^2\,. \tag{15.24}$$

2. Using the dependences (15.22), (15.23) and (15.24), we represent the displacements vector

$$\vec{\delta} = \left[u,\ \nu,\ \frac{d\nu}{dx} \right]^T$$

in the following form:

$$\vec{\delta} = L\,\vec{a}, \qquad (15.25)$$

where L is matrix of coefficients:

$$L = \begin{array}{|c|c|c|c|c|c|} \hline 1 & & & x & & \\ \hline & 1 & x & & x^2 & x^3 \\ \hline & & 1 & & 2x & 3x^2 \\ \hline \end{array} \qquad (15.26)$$

$\vec{a}$ is the vector of unknown parameters:

$$\vec{a} = \left[a_1\ a_2\ a_3\ a_4\ a_5\ a_6 \right]^T .$$

3. For the edge cross-sections ($x = 0$, $x = l$) of the bar, using expressions (15.25) and (15.26), we obtain:

$$\begin{bmatrix} Z_1' \\ Z_2' \\ Z_3' \\ Z_4' \\ Z_5' \\ Z_6' \end{bmatrix} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & & & & & \\ \hline & 1 & & & & \\ \hline & & 1 & & & \\ \hline 1 & & & l & & \\ \hline & 1 & l & & l^2 & l^3 \\ \hline & & 1 & & 2l & 3l^2 \\ \hline \end{array} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

or

$$\vec{Z}' = H\,\vec{a}\,, \qquad (15.27)$$

where H is the coupling matrix.

4. From (15.27) it follows that:

$$\vec{a} = H^{-1}\,\vec{Z}', \tag{15.28}$$

in this case

$$H^{-1} = \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline & 1 & & & & \\ \hline & & 1 & & & \\ \hline -1/l & & & 1/l & & \\ \hline & -3/l^2 & -2/l & & 3/l^2 & -1/l \\ \hline & 2/l^3 & 1/l^2 & & -2/l^3 & 1/l^2 \\ \hline \end{array}\,.$$

5. The displacement vector $\vec{\delta}$ using dependencies (15.25) and (15.28) is represented in the form:

$$\vec{\delta} = L\,H^{-1}\,\vec{Z}'\,. \tag{15.29}$$

Performing matrix multiplication, we obtain:

$$u = \left(1 - \frac{x}{l}\right) Z_1' + \frac{x}{l} Z_4'\,,$$

$$\nu = \left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right) Z_2' + \left(x - \frac{2x^2}{l} + \frac{x^3}{l^2}\right) Z_3' + \\ + \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right) Z_5' + \left(-\frac{x^2}{l} + \frac{x^3}{l^2}\right) Z_6',$$

$$\frac{d\nu}{dx} = \left(-\frac{6x}{l^2} + \frac{6x^2}{l^3}\right) Z_2' + \left(1 - \frac{4x}{l} + \frac{3x^2}{l^2}\right) Z_3' + \\ + \left(\frac{6x}{l^2} - \frac{6x^2}{l^3}\right) Z_5' + \left(-\frac{2x}{l} + \frac{3x^2}{l^2}\right) Z_6'.$$

The expressions for u and ν coincide with those recorded earlier in section 15.7.

Thus, using (15.22) and the approximating polynomial (15.23), exact functions are obtained that allow one to calculate the horizontal displacement, deflection, and rotation angle of any cross-section of the bar.

6. Nodal displacements Z_1', Z_2', Z_3' and Z_4', Z_5', Z_6' of the bar AB correspond to the reactions R_1', R_2', R_3' and R_4', R_5', R_6', allowing to find the forces N, Q and M in the edge cross-sections. To determine them, we use differential dependencies:

$$\frac{N}{EA} = \frac{du}{dx}, \qquad \frac{Q}{EJ} = \frac{d^3\nu}{dx^3}, \qquad \frac{M}{EJ} = \frac{d^2\nu}{dx^2}.$$

Differentiating in the expression (15.25) the first and third rows one time, and the second row – three times, we find the components of the vector

$$\vec{k} = \left[\frac{du}{dx}, \frac{d^3\nu}{dx^3}, \frac{d^2\nu}{dx^2} \right]^T,$$

with help of which the forces N, Q and M in the intermediate cross-sections of the bar are determined:

$$\vec{k} = B\, H^{-1}\, \vec{Z}, \tag{15.30}$$

where

$$B = \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & & \\ \hline & & & & & 6 \\ \hline & & & & 2 & 6x \\ \hline \end{array}\,.$$

To determine the forces in the edge cross-sections of the bar, we form a matrix B_{AB}, the first three rows of which correspond to the matrix B for $x = 0$, and the rest – for $x = l$.

$$B_{AB} = \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & & \\ \hline & & & & & 6 \\ \hline & & & & 2 & \\ \hline & & & 1 & & \\ \hline & & & & & 6 \\ \hline & & & & 2 & 6l \\ \hline \end{array} .$$

Then the internal forces vector

$$\vec{S} = \left[N_H, Q_H, M_H, N_K, Q_K, M_K \right]^T$$

can be calculated using the expression:

$$\vec{S}_{AB} = D\, B_{AB}\, H^{-1}\, \vec{Z}', \tag{15.31}$$

where D is the diagonal stiffness matrix:

$$D = diag\left[EA, EJ, EJ, EA, EJ, EJ \right].$$

7. The directions of positive efforts N, Q and M in the edge cross-sections of the bars (vector $\vec{S}_{AB}$) and positive reactions R_1', R_2', R_3, R_2, R_5', R_6' (their directions coincide with the directions of the components of the vector $\vec{S}$) do not coincide. The relationship between them can be established using correspondence matrix of the efforts signs by the expression:

$$\vec{R}' = I\, \vec{S}_{AB}, \tag{15.32}$$

where $$I = diag\left[-1, 1, -1, 1, -1, 1 \right].$$

Substituting the expression (15.31) in (15.32), we obtain:

$$\vec{R}' = I\, D\, B_{AB}\, H^{-1}\, \vec{Z}'. \tag{15.33}$$

From (15.33) it follows that the reactions matrix is determined by the expression:

$$R' = I\ D\ B_{AB}\ H^{-1}\,. \tag{15.34}$$

In the case under consideration, the reactions matrix (which is also the stiffness matrix) has the form shown earlier in formula (15.21).

The second method to obtain the stiffness matrix of the bar, based on the use of the Lagrange principle, is as follows.

With the well-known equation of the bended axis (see section 15.7) of the bar, its rigidity matrix can be obtained from the total energy stationary condition.

Let us write the expression of the total energy for a bar clamped (fixed) at its ends and loaded with a distributed load:

$$\begin{aligned}E &= \int_0^l \left[\frac{EJ\nu''^2}{2} + \frac{EA\,u'^2}{2} - q(x)\,\nu \right] dx = \\ &= \int_0^l \left[\frac{EJ}{2} \left(\frac{d^2\left(Z_2' f_2 + Z_3' f_3 + Z_5' f_5 + Z_6' f_6 \right)}{dx^2} \right)^2 + \right. \\ &\quad + \frac{EA}{2} \left(\frac{d\left(Z_1' f_1 + Z_4' f_4 \right)}{dx} \right)^2 - \\ &\quad \left. - q(x)\left(Z_2' f_2 + Z_3' f_3 + Z_5' f_5 + Z_6' f_6 \right) \right] dx,\end{aligned} \tag{15.35}$$

where f_1, f_4 are the basis functions for determining the longitudinal displacements of the bar cross-sections;

f_2, f_3 f_5, f_6 are functions determining the deflections of the bar (table. 15.1).

After the necessary transformations, we obtain:

$$E\left(Z_1', Z_2', Z_3', Z_4', Z_5', Z_6' \right) =$$

$$= \int_0^l \left[\frac{EJ}{2} \left(Z_2'^2 \frac{12}{l^3} + Z_3'^2 \frac{4}{l} + Z_5'^2 \frac{12}{l^3} + Z_6'^2 \frac{4}{l} + 2 Z_2' Z_3' \frac{6}{l^2} - \right. \right.$$

$$-2 Z_2' Z_5' \frac{12}{l^3} + 2 Z_2' Z_6' \frac{6}{l^2} - 2 Z_3' Z_5' \frac{6}{l^2} + 2 Z_3' Z_6' \frac{2}{l} - \qquad (15.36)$$

$$\left. -2 Z_5' Z_6' \frac{6}{l^2} \right) + \frac{EA}{2l^2} \left(Z_1'^2 - 2 Z_1' Z_4' + Z_4'^2 \right) -$$

$$\left. -q(x) \left(Z_2' f_2 + Z_3' f_3 + Z_5' f_5 + Z_6' f_6 \right) \right] dx.$$

In this expression, the total energy is represented by a function of six independent variables.

Integrals of the form

$$\int_0^l q(x) f_i dx$$

give us the values of the support reactions for the beam pinched at the ends when loading it with a distributed load $q(x)$. To prove this statement, we consider the same beam (Figure 15.13) in two states: in the state a, the beam is loaded with a distributed load $q(x)$, in the state b the left end of the beam is moved by $Z_2' = 1$. Remember that the positive directions of reactions correspond to the positive directions of displacements.

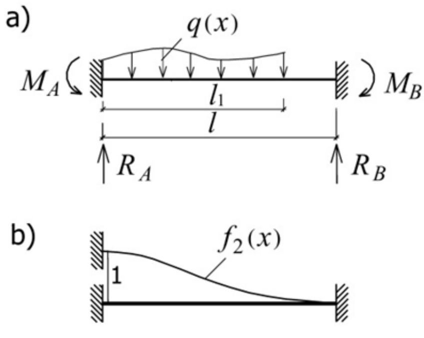


Figure 15.13

The virtual work of the forces of state *a* on the displacements of state *b* is:

$$W_{aб} = R_A \cdot 1 - \int_0^{l_1} q(x)\left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right) dx .$$

The work of the forces of state *b* on the displacements of state *a* is zero:

$$W_{ba} = 0 .$$

Based on the reciprocity theorem:

$$W_{ab} = W_{ba} .$$

Hence,

$$R_A = \int_0^{l_1} q(x)\left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right) dx .$$

When calculating M_A for an auxiliary state, we will take what is shown in table. 15.1 (see section 2). Having determined the possible work of the forces of one state on the displacements of another (in the forward and reverse directions), based on the reciprocity theorem, we obtain:

$$M_A = \int_0^{l_1} q(x)\left(x - \frac{2x^2}{l} + \frac{x^3}{l^2}\right) dx .$$

We find by similar calculations:

$$R_B = \int_0^{l_1} q(x)\left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right) dx ,$$

$$M_B = -\int_0^{l_1} q(x)\left(-\frac{x^2}{l} + \frac{x^3}{l^2}\right) dx .$$

Note that when a different type of load (concentrated forces, moments, etc.) is applied to the bar, the support reactions can also be calculated using the reciprocity theorem.

The necessary conditions for the minimum function (15.36) of six variables are:

$$\frac{\partial E}{\partial Z_i'} = 0, \qquad i = \overline{1, 6}.$$

Applying them to the expression (15.36), we obtain a system of equations relating the displacements and reactions of the edge cross-sections:

$$\begin{cases}
\frac{EA}{l} Z_1' & 0 & 0 & -\frac{EA}{l} Z_4' & 0 & 0 & = 0; \\
0 & \frac{12EJ}{l^3} Z_2' & \frac{6EJ}{l^2} Z_3' & 0 & -\frac{12EJ}{l^3} Z_5' & \frac{6EJ}{l^2} Z_6' & = R_A; \\
0 & \frac{6EJ}{l^2} Z_2' & \frac{4EJ}{l} Z_3' & 0 & -\frac{6EJ}{l^2} Z_5' & \frac{2EJ}{l} Z_6' & = M_A; \\
-\frac{EA}{l} Z_1' & 0 & 0 & \frac{EA}{l} Z_4' & 0 & 0 & = 0; \\
0 & -\frac{12EJ}{l^3} Z_2' & -\frac{6EJ}{l^2} Z_3' & 0 & \frac{12EJ}{l^3} Z_5' & -\frac{6EJ}{l^2} Z_6' & = R_B; \\
0 & \frac{6EJ}{l^2} Z_2' & \frac{2EJ}{l} Z_3' & 0 & -\frac{6EJ}{l^2} Z_5' & \frac{4EJ}{l} Z_6' & = -M_B.
\end{cases}$$

In matrix form, the system has the form:

$$R' \vec{Z}' = \vec{F}',$$

where R' is the stiffness matrix of the bar (15.21);

$\vec{Z}' = [Z_1', Z_2', Z_3', Z_4', Z_5', Z_6']^T$ is the vector of nodal displacements;

$\vec{F}' = [0, R_A, M_A, 0, R_B, -M_B]^T$ is the vector of nodal forces.

Another way to solve this problem is based on the general equations of structural mechanics. Given the initial conditions, we write the bar equilibrium matrix in the form:

$$a = \begin{bmatrix} -\cos\varphi & \dfrac{\sin\varphi}{l} & -\dfrac{\sin\varphi}{l} \\ -\sin\varphi & -\dfrac{\cos\varphi}{l} & \dfrac{\cos\varphi}{l} \\ 0 & -1 & 0 \\ \cos\varphi & -\dfrac{\sin\varphi}{l} & \dfrac{\sin\varphi}{l} \\ \sin\varphi & \dfrac{\cos\varphi}{l} & -\dfrac{\cos\varphi}{l} \\ 0 & 0 & 1 \end{bmatrix}.$$

Then for $\varphi = 0$ we get:

$$R' = a\,k\,a^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\dfrac{1}{l} & \dfrac{1}{l} \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & \dfrac{1}{l} & -\dfrac{1}{l} \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \dfrac{EA}{l} & 0 & 0 \\ 0 & \dfrac{4EI}{l} & \dfrac{-2EI}{l} \\ 0 & \dfrac{-2EI}{l} & \dfrac{4EI}{l} \end{bmatrix} \times$$

$$\times \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\dfrac{1}{l} & -1 & 0 & \dfrac{1}{l} & 0 \\ 0 & \dfrac{1}{l} & 0 & 0 & -\dfrac{1}{l} & 1 \end{bmatrix} =$$

$$
= \begin{bmatrix}
\frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\
0 & \frac{12EJ}{l^3} & \frac{6EJ}{l^2} & 0 & -\frac{12EJ}{l^3} & \frac{6EJ}{l^2} \\
0 & \frac{6EJ}{l^2} & \frac{4EJ}{l} & 0 & -\frac{6EJ}{l^2} & \frac{2EJ}{l} \\
-\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\
0 & -\frac{12EJ}{l^3} & -\frac{6EJ}{l^2} & 0 & \frac{12EJ}{l^3} & -\frac{6EJ}{l^2} \\
0 & \frac{6EJ}{l^2} & \frac{2EJ}{l} & 0 & -\frac{6EJ}{l^2} & \frac{4EJ}{l}
\end{bmatrix}.
$$

The dimensions of the matrix R' depend on the conditions for fixing the ends of the bar. For a bar with one pinched (fixed) end, and second hinge-supported end, the stiffness matrix will have dimensions 5×5, for a bar with a hinged support on the left and right ends, the matrix R' has dimensions 4×4.

The stiffness matrix of the bar connecting the nodes i and j, can be represented in block form:

$$
R' = \begin{bmatrix} R'_{ii} & R'_{ij} \\ R'_{ji} & R'_{jj} \end{bmatrix}.
$$

The sizes of the blocks depend on the number of constraints superimposed on the bar in each node. For a bar hinge-supported in node i and rigidly-pinched in the node j, the dimensions of the matrix and its blocks will be as follows:

$$
R'_{(5\times5)} = \begin{bmatrix} R'_{ii_{(2\times2)}} & R'_{ij_{(2\times3)}} \\ R'_{ji_{(3\times2)}} & R'_{jj_{(3\times3)}} \end{bmatrix}.
$$

If both ends of the bar are hinge-supported, then:

$$R'_{(4\times4)} = \begin{bmatrix} R'_{ii_{(2\times2)}} & R'_{ij_{(2\times2)}} \\ R'_{ji_{(2\times2)}} & R'_{jj_{(2\times2)}} \end{bmatrix}.$$

15.10. Bar Stiffness Matrix in the General Coordinate System

The bar stiffness matrix can be obtained the easiest way using the general equations of structural mechanics by the expression:

$$R = a\,k\,a^T\,,$$

where a is the equilibrium matrix of the bar in the general coordinate system. We show the formation of the stiffness matrix for the truss rod (1–2) of the truss shown in Figure 15.14.

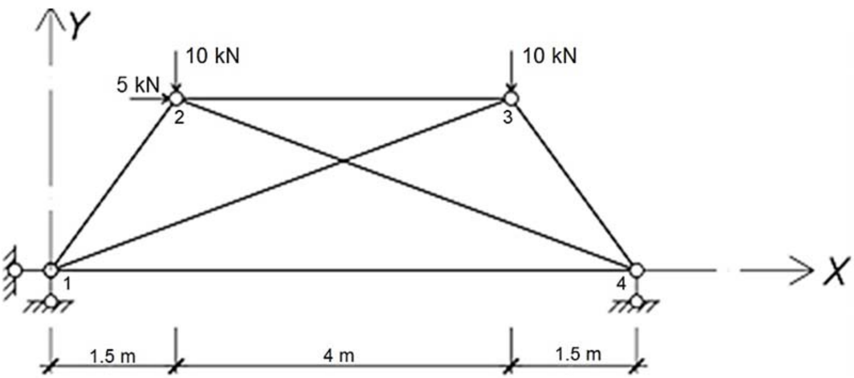


Figure 15.14

The rod is adjacent to the hinge nodes. We introduce the links at the rod ends that impede their displacements along the directions of the coordinate axes (Figure 15.15, a). We show in the same figure the positive directions of the nodes displacements.

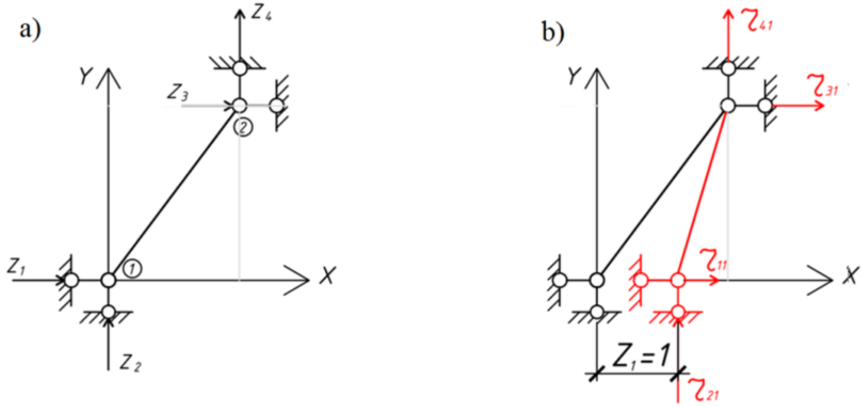


Figure 15.15

The stiffness matrix for the rod (1–2) has the form:

$$R^{(1-2)} = \begin{array}{|c|c|c|c|} \hline r_{11} & r_{12} & r_{13} & r_{14} \\ \hline r_{21} & r_{22} & r_{23} & r_{24} \\ \hline r_{31} & r_{32} & r_{33} & r_{34} \\ \hline r_{41} & r_{42} & r_{43} & r_{44} \\ \hline \end{array}$$

Figure 15.15, b shows the displacement of node 1 in the direction of Z_1 and indicates the positive directions of the reactions in the introduced links.

We calculate $R^{(1-2)}$ and present the result in block form.

$$R^{(1-2)} = a_{1-2} \cdot k_{1-2} \cdot a_{1-2}^{T}; \quad a_{1-2}^{T} = [-0.6, -0.8, 0.6, 0.8],$$

$$k_{1-2} = 2EA/2.5.$$

$$R^{(1-2)} = \begin{array}{|c|c|c|c|} \hline 0.288 & 0.384 & -0.288 & -0.384 \\ \hline 0.384 & 0.512 & -0.384 & -0.512 \\ \hline -0.288 & -0.384 & 0.288 & 0.384 \\ \hline -0.384 & -0.512 & 0.384 & 0.512 \\ \hline \end{array} \cdot EI = \begin{array}{|c|c|} \hline R_{11}^{(1-2)} & R_{12}^{(1-2)} \\ \hline R_{21}^{(1-2)} & R_{22}^{(1-2)} \\ \hline \end{array}$$

The reaction values in additional links caused by the displacement $Z=1$ are given in the first column of the matrix $R^{(1-2)}$.

In other cases, when it is necessary to organize the transition from the stiffness matrix in the local coordinate system to the stiffness matrix in the general coordinate system, it is necessary to use the transformation rules of the linear operator matrix in the transition from the old basis to the new one.

Linear displacements in the local and general coordinate systems (Figure 15.16) are related by the relations:

$$Z_1' = Z_1 \cos\varphi + Z_2 \sin\varphi,$$

$$Z_2' = -Z_1 \sin\varphi + Z_2 \cos\varphi.$$

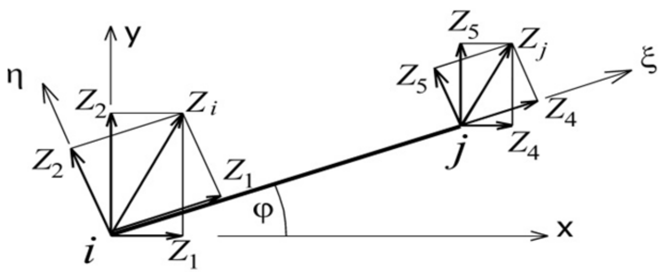


Figure 15.16

The rotation angle of the bar end section does not change when the coordinate system is changed. Therefore, the matrix of the rotation operator has the form:

$$C = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the directions of displacements at both ends of the bar coincide, transformations of the displacements Z_4', Z_5' and Z_6' to Z_4, Z_5 and Z_6 are performed using the same matrix. Consequently,

$$\vec{Z}' = \begin{bmatrix} C & \\ & C \end{bmatrix} \vec{Z} = V\,\vec{Z}\,. \tag{15.41}$$

Since $\quad V^{-1} = V^T, \quad$ then $\quad \vec{Z} = V^T\,\vec{Z}'\,.$

The transformation of the stiffness matrix (matrix of the linear operator) during the transition to the basis $\vec{Z}$ is performed by the expression:

$$R = V^T R' V\,.$$

For further discussion, we also present the matrix R in block form:

$$R = \begin{bmatrix} R_{ii} & R_{ij} \\ R_{ji} & R_{jj} \end{bmatrix}.$$

15.11. Stiffness Matrix Formation for the Entire System

The finite element model of the bars system, as already noted, is represented as a set of bars connected in nodes. The system node displacements cause the same end displacements of the bars (finite elements) adjacent to this node. The resulting forces in the bars of the primary system of the displacement method are determined using the stiffness matrices of the bars. The reactions in the links superimposed on the node of the system can be found as the sum of the terminal reactions in the links of the bars adjacent to the node. For example, the reaction force in the link along the X axis direction will be equal to the sum of the reaction forces in the links of the bars in the same direction. Reactions in other directions are defined in a similar way.

In the general case, the vector $\vec{R}_i$ of total reactions for the i-th node of the system can be determined through the vectors $\vec{r}_i^{(e)}$ of end reactions in elements adjacent to this node, by expression:

$$\begin{aligned} \vec{R}_i = \sum_{e\in i} \vec{r}_i^{(e)} = \sum_{e\in i} \vec{r}_{i1}^{(e)}\, Z_1 + \sum_{e\in i} \vec{r}_{i2}^{(e)}\, Z_2 + \cdots + \\ + \sum_{e\in i} \vec{r}_{ii}^{(e)}\, Z_i + \sum_{e\in i} \vec{r}_{ij}^{(e)}\, Z_j + \cdots + \sum_{e\in i} \vec{r}_{in}^{(e)}\, Z_n, \end{aligned} \tag{15.42}$$

where $\vec{r}_{i1}^{(e)}$, $\vec{r}_{i2}^{(e)},\cdots$, $\vec{r}_{in}^{(e)}$ – reaction vectors at the end of an element adjacent to a node caused by displacements $Z_1 = 1$, $Z_2 = 1,\cdots, Z_n = 1$. Symbol $e \in i$ means summation over all elements adjacent to the node i.

For a rigid node, vector $\vec{R}_i$ has three components: the first component of the vector indicates the value of the reactive force in the direction of the axis X, the second – in the direction of the axis Y, the third gives the value of the reactive moment.

If all the bars are connected at nodes rigidly, the vectors $\vec{r}_{i1}$, $\vec{r}_{i2}$, $\cdots$, $\vec{r}_{in}$ also contain three components. If some bar adjoins the node articulated, then for the operation of adding vectors in the *i*-th node, the third component of the end reaction vector should be taken equal to zero.

Having written expression (15.42) for each node of the structure, we present a system of equations connecting nodal reactions and displacements in the form:

$$\begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_m \end{bmatrix} = \begin{bmatrix} \sum\limits_{e\in 1} r_{11}^{(e)} & \sum\limits_{e\in 1} r_{12}^{(e)} & \cdots & \sum\limits_{e\in 1} r_{1j}^{(e)} & \cdots & \sum\limits_{e\in 1} r_{1n}^{(e)} \\ . & . & \cdots & . & \cdots & . \\ \sum\limits_{e\in i} r_{i1}^{(e)} & \sum\limits_{e\in i} r_{i2}^{(e)} & \cdots & \sum\limits_{e\in i} r_{ij}^{(e)} & \cdots & \sum\limits_{e\in i} r_{in}^{(e)} \\ . & . & \cdots & . & \cdots & . \\ \sum\limits_{e\in m} r_{m1}^{(e)} & \sum\limits_{e\in m} r_{m2}^{(e)} & \cdots & \sum\limits_{e\in m} r_{mj}^{(e)} & \cdots & \sum\limits_{e\in m} r_{mn}^{(e)} \end{bmatrix} \cdot \begin{bmatrix} Z_1 \\ \vdots \\ Z_i \\ \vdots \\ Z_m \end{bmatrix}$$

or abbreviated:

$$\vec{R}_z = R\,\vec{Z}\,,$$

where R is the stiffness matrix of the entire system.

From the presented form of recording the matrix R it follows that its elements are calculated through the elements of the stiffness matrices of individual finite elements. If the nodes i and j are not interconnected by elements, then $r_{ij} = 0$; if they are connected by several elements, then the corresponding element of the stiffness matrix is calculated as

$$r_{ij} = \sum_{e \in i,j} r_{ij}^{(e)}.$$

In block form, the matrix R is represented as:

$$R = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ . & . & \cdots & . \\ R_{i1} & R_{i2} & \cdots & R_{in} \\ . & . & \cdots & . \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{bmatrix},$$

where R_{ij} is the reaction block in the links of the i-th node caused by the unit displacements of the links of the j-th node.

In the stiffness matrix of the entire system formed according to the indicated principle, support nodes had not been taken into account. The displacements of the non-deformable supports are equal to zero. Therefore, if it is known in advance that $Z_j = 0$, then the j-th row and the j-th column of the obtained matrix R should be deleted. The size of the matrix must be reduced. In the case of automated computing, a new numbering of unknowns will also be required. If the dimensions of the matrix will not be changed, then it is necessary to take the indicated row and column as zero, but the element r_{jj} of the matrix R must be taken equal to one or to a number other than zero (so that $\det R \neq 0$).

We will show a graphical scheme of stiffness matrix formation of the entire system from the stiffness matrices of its elements.

For the frame shown in figure 15.17, the stiffness matrix in block form is written as:

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}.$$

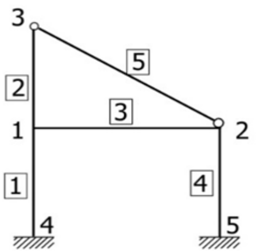


Figure 15.17

The contribution of each of the five elements (numbers are written in squares) to the corresponding blocks of the stiffness matrix of the entire frame is shown schematically in Figure 15.18, a – e. The stiffness matrix of the entire frame is shown in Figure 15.18, f.

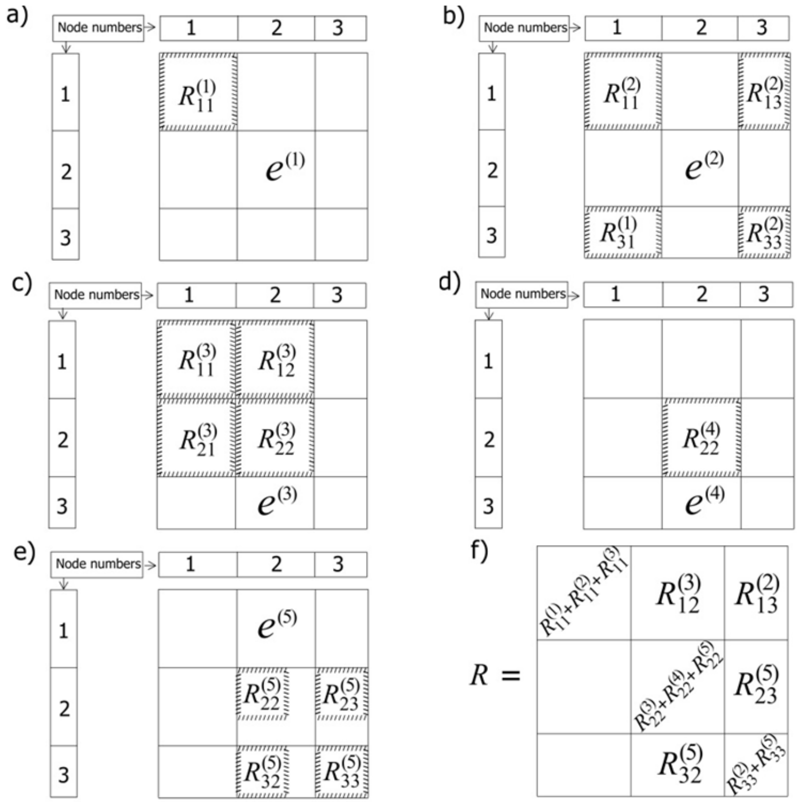


Figure 15.18

The numbers of degrees of freedom for each node of the frame correspond to the numbers of the components of the displacement vector for this node (1 – displacement along the axis X , 2 – displacement along the axis Y , 3 – rotation about the axis Z).

In Figures 15.18 symbol $e^{(i)}$ is the number of the corresponding bar element.

15.12. Stiffness Matrix of a Rectangular Finite Element for Calculating Thin Plates

The displacements of the finit element must correspond to the deformed scheme of the system under study at its location. In the general case, it is usually impossible to accurately describe the state of a continuum system through a finite set of nodal displacements using equations of type (15.18). Therefore, the FEM is classified as approximate. Nevertheless, it allows obtaining the calculation results of very high accuracy. Currently, FEM is the main method for solving the most diverse problems of statics, dynamics and stability of bars and continuum systems.

The procedure for obtaining stiffness matrices of FE for calculating plates, shells, and other continuum systems is in many respects similar to the method for obtaining a bar stiffness matrix (see Section 15.11). Let us explain this note by the example of constructing the FE stiffness matrix for plate calculation.

Brief information from the theory of plate calculation

A plate is a body whose thickness h is small compared with the dimensions of the sides of the base a and b (Figure 15.19, a).
The plane dividing the thickness of the plate in half is called the median. The intersection lines of the median plane with the lateral surface form the contour of the plate. According to the shape in plan, plates are distinguished rectangular, triangular, round, etc.

When calculating the plates, the origin of the coordinate axes is located in one of the points of the median plane. From the action of the transverse load, the plate bends, the median plane turns into the median surface. The displacements of the plate points in the direction of the axes x , y , z are denoted by u , v , w respectively.

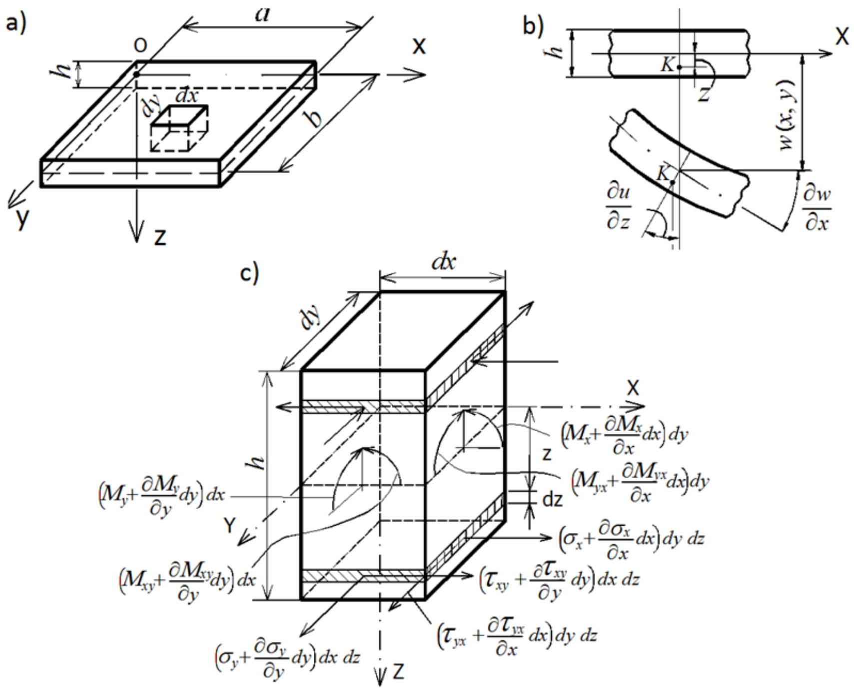


Figure 15.19

In general, these displacements are functions of coordinates:

$$u = u(x, y, z), \; v = v(x, y, z), \; w = w(x, y, z).$$

Depending on the nature of the stress state of the plates, they are divided into thick plates (the ratio of the thickness h to the larger of the dimensions is greater than 0.10…0.20), thin plates (the corresponding ratio is in the range from 0.01 up to 0.10), very thin plates (the ratio less than 0.01).

The "classical" theory of plates is applicable to very thin and moderately thin plates.

For the thick plates it becomes erroneous to view such structural element as a plate – a description based on the three-dimensional theory of elasticity is required.

In turn, thin plates are divided into rigid thin plates (those for which tensile and shear forces in the middle surface are not taken into account during bending) and flexible thin plates (the shear and tensile stresses in the middle surface are taken into account during bending).

The theory of calculating thin plates is constructed using the following hypotheses.

1. The direct normals hypothesis, according to which a rectilinear element normal to the median plane before deformation of the plate remains normal to the median plane after deformation, and its length does not change. According with this hypothesis, the shear angles γ_{xz} and γ_{yz}, as well as the linear deformation ε_z are taken equal to zero:

$$\gamma_{xz} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0\,; \quad \gamma_{yz} = \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0\,; \quad \varepsilon_z = 0\,.$$

2. The hypothesis of deformability of the middle layer, according to which the linear ε_x, ε_y and angular γ_{xy} deformations of the middle layer are equal to zero:

$$\varepsilon_x^0 = \left(\frac{\partial u}{\partial x} \right)_{z=0} = 0\,; \qquad \varepsilon_y^0 = \left(\frac{\partial v}{\partial y} \right)_{z=0} = 0;$$

$$\gamma_{xy}^0 = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)_{z=0} = 0\,.$$

3. The hypothesis of the absence of normal stresses on sites parallel to the middle layer, that is, stress $\sigma_z = 0$.

In accordance with the first two hypotheses, the displacements u and v of an arbitrary point (Figure 15.19, b) along the directions of the axes x and y are equal

$$u = -z\,\frac{\partial w}{\partial x}\,, \qquad v = -z\,\frac{\partial w}{\partial y}\,.$$

As a result, linear and angular deformations are calculated by the formulas:

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}, \qquad \varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2},$$

$$\gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -2z\frac{\partial^2 w}{\partial x \partial y}.$$

Since $\sigma_z = 0$, the generalized Hooke law, connecting stresses and strains, is written as:

$$\sigma_x = \frac{E}{1-\mu^2}\left(\varepsilon_x + \mu\varepsilon_y\right), \qquad \sigma_y = \frac{E}{1-\mu^2}\left(\varepsilon_y + \mu\varepsilon_x\right),$$

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\mu)}\gamma_{xy},$$

where μ is the Poisson ratio.

The normal and tangential stresses caused by the bending of the plate linearly vary along the thickness of the plate and are calculated through the curvature $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$ and torsion $\frac{\partial^2 w}{\partial x \partial y}$ of the middle surface according to the formulas:

$$\sigma_x = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right),$$

$$\sigma_y = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right), \qquad \tau_{xy} = -\frac{Ez}{1+\mu}\frac{\partial^2 w}{\partial x \partial y}.$$

Bending moments M_x and M_y and torque M_{xy} per unit length of the plate section are calculated through the corresponding stresses.

Figure 15.19, c shows the distribution of forces along the faces of an elementary prism $dx \times dy \times h$. In order not to clutter the drawing, stresses, bending moments and torques are not shown on the faces $x = 0$ and $y = 0$. On these faces, the increments of stresses are equal to zero

15.13. Stiffness Matrix Formation Example of the Rectangular Plate Element

In this example, the finite element of a rigid plate is used.

1. Each node of the plate finite element has three degrees of freedom:

w – vertical displacement (deflection),

$\dfrac{\partial w}{\partial y}$ – angle of rotation about the axis x,

$\dfrac{\partial w}{\partial x}$ – angle of rotation about the axis y.

In the directions of these displacements, we impose additional links and thus obtain the primary system of the displacement method (Figure 15.20).

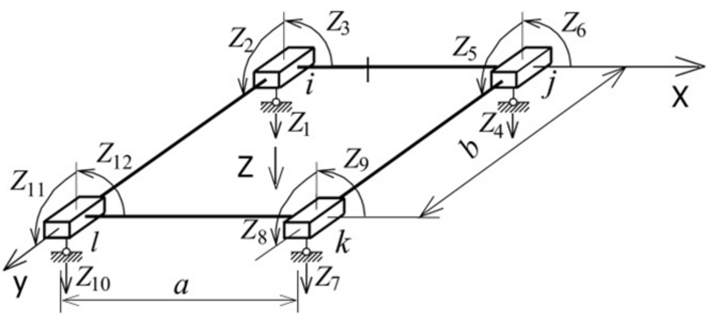


Figure 15.20

It is necessary to form a matrix in the local coordinate system, which would make it possible to transform the vector of nodal displacements $\vec{Z}$ into vector of nodal reactions $\vec{R}_Z$.

We define the deflections function $w(x, y)$ of the element in the form of a polynomial with 12 arbitrary constants. It must identically satisfy the homogeneous (load acts in nodes) differential equation of the deformed plate surface:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^2 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = 0 .$$

Let us assume that:

$$\begin{aligned} w(x,y) = a_1 + a_2\, x + a_3\, y + a_4\, x^2 + a_5\, x\, y + a_6\, y^2 + a_7\, x^3 \\ + a_8\, x^2\, y + a_9\, x\, y^2 + a_{10}\, y^3 + a_{11}\, x^3\, y + a_{12}\, x\, y^3, \end{aligned} \tag{15.43}$$

where a_i – unknown independent parameters, which in the future must be expressed in terms of $\vec{Z}$.

2. Angular displacements $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are determined uniquely by the expression $w(x,y)$. Then for any point of the element the displacement vector can be determined by the dependence:

$$\vec{u} = L\,\vec{a}\,, \tag{15.44}$$

where $$\vec{u} = \left[w, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial x} \right]^T ;$$

L – coefficient matrix of dimension 3 by 12:

	1	*x*	*y*	x^2	*xy*	y^2	x^3	x^2y	xy^2	y^3	x^3y	xy^3
L=			*1*		*x*	*2y*		x^2	*2xy*	$3y^2$	x^3	$3xy^2$
		-1		*-2x*	*-y*		$-3x^2$	*-2xy*	$-y^2$		$-3xy^2$	$-y^3$

3. The displacements vector $\vec{u}$ allows us to find the displacements of all element points , including nodal ones, having coordinates ($x=0$, $y=0$), ($x=a$, $y=0$), ($x=a$, $y=b$), ($x=0$, $y=b$). Therefore, using the expression (15.44), we can establish the relationship between the vectors $\vec{Z}$ and $\vec{a}$:

$$\vec{Z} = H\,\vec{a}\,, \tag{15.46}$$

where H is the transformation matrix of dimension 12 by 12:

H =

1											
		1									
	-1										
1	a		a^2			a^3					
		1		a			a^2			a^3	
	-1		-2a			$-3a^2$					
1	a	b	a^2	ab	b^2	a^3	a^2b	ab^2	b^3	a^3b	ab^3
		1		a	2b		a^2	2ab	$3b^2$	a^3	$3ab^2$
	-1		-2a	-b		$-3a^2$	-2ab	$-b^2$		$-3a^2b$	$-b^3$
1		b			b^2				b^3		
		1			2b				$3b^2$		
	-1			-b				$-b^2$			$-b^3$

4. From (15.46) it follows that:

$$\vec{a} = H^{-1}\,\vec{Z}\,. \tag{15.47}$$

5. The displacements vector $\vec{u}$ using expressions (15.44) and (15.47) is represented in the form:

$$\vec{u} = L\,H^{-1}\,\vec{Z}. \tag{15.48}$$

6. After determining the displacement vector, one can find the vector of generalized relative deformations $\vec{k}$, whose components are the curvature and torsion of the middle surface of the plate:

$$\vec{k} = \left[\frac{\partial^2 w}{\partial x^2}, \quad \frac{\partial^2 w}{\partial y^2}, \quad 2\frac{\partial^2 w}{\partial x\,\partial y} \right]^T . \tag{15.49}$$

Performing the appropriate differentiation, we obtain:

$$\vec{k} = B\,\vec{a} = B\,H^{-1}\;\vec{Z}\,, \tag{15.50}$$

where

$B=$

			2			6x	2y			6xy	
					2			2x	6y		6xy
				2			4x	4y		$6x^2$	$6y^2$

7. Linearly distributed (per unit length of the plate section) bending moments and torques for isotropic plates are calculated by the formulas:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\,\frac{\partial^2 w}{\partial y^2}\right),$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\,\frac{\partial^2 w}{\partial x^2}\right), \qquad (15.51)$$

$$M_{xy} = -D\,(1-\mu\,)\,\frac{\partial^2 w}{\partial x \partial y},$$

where D denotes the value of linear bending stiffness of the plate, the so-called cylindrical stiffness:

$$D = \frac{E\,h^3}{12\left(1-\mu^2\right)}.$$

Bending moments corresponding to positive curvatures are considered positive.

In the matrix form of notation, the relationship of the vector of generalized internal forces $\vec{M}$ with the vector of relative strains $\vec{k}$ takes the form:

$$\vec{M} = C\,\vec{k}\,, \qquad (15.52)$$

where C is the matrix of physical constants:

$$C = D\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \dfrac{1-\mu}{2} \end{bmatrix}.$$

The expression of the vector of generalized internal forces $\vec{M}$ through generalized displacements $\vec{Z}$ can be obtained by substituting expression $\vec{k}$ from (15.50) in the dependence (15.52):

$$\vec{M} = C\,B\,H^{-1}\,\vec{Z}\,. \tag{15.53}$$

8. Variation of the potential energy density of the element deformation

$$\delta\,\overline{A} = \left(\delta\vec{k}\right)^T\ \vec{M}\,, \tag{15.54}$$

taking into account expressions (15.50) and (15.53), can be written as follows:

$$\delta\,\overline{A} = \left(\delta\vec{Z}\right)^T\ \left(H^{-1}\right)^T\ B^T C\,B\,H^{-1}\,\vec{Z}\,.$$

For the entire volume of the finite element, the variation of the potential strain energy will have the form:

$$\delta A = \int\limits_V \delta\,\overline{A} dv = \int\limits_v \left(\delta\vec{Z}\right)^T\ \left(H^{-1}\right)^T\ B^T C\,B\,H^{-1}\vec{Z}\,dv \tag{15.55}$$

or, since H and $\vec{Z}$ are independent of the coordinates x and y,

$$\delta A = \left(\delta\vec{Z}\right)^T\ \left(H^{-1}\right)^T\left[\int\limits_v B^T\ C\,B\,dv\right]H^{-1}\,\vec{Z}\,. \tag{15.56}$$

9. Virtual work of nodal forces F on changes (variations) of nodal displacements $\delta\vec{Z}$ is:

$$\delta W = \left(\delta\vec{Z}\right)^T \vec{F} . \qquad (15.57)$$

In accordance with the principle of virtual displacements $\delta A = \delta W$, therefore equalities (15.55) and (15.56) allow us to relate the vector of nodal forces $\vec{F}$ and the vector $\vec{Z}$:

$$\vec{F} = \left(H^{-1}\right)^T \left[\int_v B^T \, C \, B \, dv\right] H^{-1} \, \vec{Z} . \qquad (15.58)$$

10. The stiffness matrix (reaction matrix) of the finite element R_E allows us to express the force vector $\vec{F}$ through the vector $\vec{Z}$:

$$\vec{F} = R_E \vec{Z} . \qquad (15.59)$$

Comparing expressions (15.58) and (15.59), we find the stiffness matrix of the finite element:

$$R_E = \left(H^{-1}\right)^T \left[\int_\nu B^T C B d\nu\right] H^{-1} .$$

The matrix R_E (lower triangle) for the rectangular element of the plate is presented in table 15.2.

Table 15.2

$\boldsymbol{R}=$

	1	2	3	4
	w_i	θ_{x_i}	θ_{y_i}	w_j
1	$120\left(\beta^2+\gamma^2\right)- -24\mu+84$	S Y M M E T R I C		
2	$\begin{bmatrix}10\beta^2+\\+(1+4\mu)\end{bmatrix}6b$	$40a^2+ +8(1-\mu)b^2$		
3	$-\begin{bmatrix}10\gamma^2+\\+(1+4\mu)\end{bmatrix}6a$	$-30\mu ab$	$40b^2+ +8(1-\mu)a^2$	
4	$60\left(\beta^2-2\gamma^2\right)+ +24\mu-84$	$\begin{bmatrix}5\beta^2-\\-(1+4\mu)\end{bmatrix}6b$	$\begin{bmatrix}10\gamma^2+\\+(1-\mu)\end{bmatrix}6a$	$120\left(\beta^2+\gamma^2\right)- -24\mu+84$
5	$\begin{bmatrix}5\beta^2-\\-(1+4\mu)\end{bmatrix}6b$	$20a^2+ +8(1-\mu)b^2$	0	$\begin{bmatrix}10\beta^2+\\+(1+4\mu)\end{bmatrix}6b$
6	$-\begin{bmatrix}10\gamma^2+\\+(1-\mu)\end{bmatrix}6a$	0	$20b^2- -2(1-\mu)a^2$	$\begin{bmatrix}10\gamma^2+\\+(1+4\mu)\end{bmatrix}6a$
7	$-60\left(\gamma^2+\beta^2\right)- -24\mu+84$	$\begin{bmatrix}-5\beta^2+\\+(1-\mu)\end{bmatrix}6b$	$\begin{bmatrix}5\gamma^2-\\-(1-\mu)\end{bmatrix}6a$	$60\left(\gamma^2-2\beta^2\right)+ +24\mu-84$
8	$\begin{bmatrix}5\beta^2-\\-(1-\mu)\end{bmatrix}6b$	$10a^2+ +2(1-\mu)b^2$	0	$\begin{bmatrix}10\beta^2+\\+(1-\mu)\end{bmatrix}6b$
9	$\begin{bmatrix}-5\gamma^2+\\+(1-\mu)\end{bmatrix}6a$	0	$10b^2+ +2(1-\mu)a^2$	$\begin{bmatrix}5\gamma^2-\\-(1+4\mu)\end{bmatrix}6a$
10	$60\left(\gamma^2-2\beta^2\right)+ +24\mu-84$	$-\begin{bmatrix}10\beta^2+\\+(1-\mu)\end{bmatrix}6b$	$\begin{bmatrix}-5\gamma^2+\\+(1+4\mu)\end{bmatrix}6a$	$-60\left(\beta^2+\gamma^2\right)- -24\mu+84$
11	$\begin{bmatrix}10\beta^2+\\+(1-\mu)\end{bmatrix}6b$	$20a^2- -8(1-\mu)b^2$	0	$\begin{bmatrix}5\beta^2-\\-(1-\mu)\end{bmatrix}6b$
12	$\begin{bmatrix}-5\gamma^2+\\+(1+4\mu)\end{bmatrix}6a$	0	$20b^2- -8(1-\mu)a^2$	$\begin{bmatrix}5\gamma^2-\\-(1-\mu)\end{bmatrix}6a$

Table 15.2 (continuation)

	5	6	7	8
	θ_{x_j}	θ_{y_j}	w_k	θ_{x_k}
1				
2				
3	S Y M M E T R I C			
4				
5	$40a^2+$ $+8(1-\mu)b^2$			
6	$30\mu ab$	$40b^2+$ $+8(1-\mu)a^2$		
7	$-\begin{bmatrix}10\beta^2+\\+(1-\mu)\end{bmatrix}6b$	$\begin{bmatrix}5\gamma^2-\\-(1+4\mu)\end{bmatrix}6a$	$120\left(\beta^2+\gamma^2\right)-$ $-24\mu+84$	
8	$20a^2-$ $-2(1-\mu)b^2$	0	$-\begin{bmatrix}10\beta^2+\\+(1+4\mu)\end{bmatrix}6b$	$40a^2+$ $+8(1-\mu)b^2$
9	0	$20b^2-$ $-8(1-\mu)a^2$	$\begin{bmatrix}10\gamma^2+\\+(1+4\mu)\end{bmatrix}6a$	$-30\mu ab$
10	$\begin{bmatrix}-5\beta^2+\\+(1-\mu)\end{bmatrix}6b$	$\begin{bmatrix}-5\gamma^2+\\+(1-\mu)\end{bmatrix}6a$	$60\left(\beta^2-2\gamma^2\right)+$ $+24\mu-84$	$\begin{bmatrix}-5\beta^2+\\+(1+4\mu)\end{bmatrix}6b$
11	$10a^2+$ $+2(1-\mu)b^2$	0	$\begin{bmatrix}-5\beta^2+\\+(1+4\mu)\end{bmatrix}6b$	$20a^2-$ $-8(1-\mu)b^2$
12	0	$10b^2+$ $+2(1-\mu)a^2$	$\begin{bmatrix}10\gamma^2+\\+(1-\mu)\end{bmatrix}6a$	0

Table. 15.2 (ending)

	9	10	11	12
	θ_{y_k}	w_l	θ_{x_l}	θ_{y_l}
1				
2				
3				
4				
5				
6				
7	S Y M M E T R I C			
8				
9	$40b^2 + +8(1-\mu)a^2$			
10	$-\begin{bmatrix} 10\gamma^2 + \\ +(1-\mu) \end{bmatrix} 6a$	$120\left(\beta^2+\gamma^2\right)- -24\mu+84$		
11	0	$\begin{bmatrix} -10\beta^2 - \\ -(1+4\mu) \end{bmatrix} 6b$	$40a^2 + +8(1-\mu)b^2$	
12	$20b^2 - -2(1-\mu)a^2$	$-\begin{bmatrix} 10\gamma^2 + \\ +(1+4\mu) \end{bmatrix} 6a$	$30\mu ab$	$40b^2 + +8(1-\mu)a^2$

Note: For all elements of the matrix, the common factor $\dfrac{D}{30ab}$ is used, where D is the cylindrical stiffness; $\beta = \dfrac{a}{b}$; $\gamma = \dfrac{b}{a}$.

15.14. General Notes on the Finite Element Method

This chapter outlines only the basics of FEM. Additional information, details of finite element approximations of higher order and others, including applied aspects of the method, can be found in the numerous educational and scientific sourses.

In modern design and computing complexes (CAD softwares) designed for numerical analysis of the stress-strain state, stability and vibrations of bars and continuum structural systems, as a rule, finite element methode (FEM) is implemented. With the help of software based on FEM, a wide range of structures is analysed: flat and spatial bars systems, arbitrary plate and shell systems, frame-and-link structures of high-rise buildings, slabs on a soil foundation, multilayer structures, membranes, suspansion and cable stayed bridges, massive bodies. The analysis is carried out on static and dynamic loads.

The composition of CAD systems includes a large number of FE types: bars; quadrangular and triangular plate elements; shell FE (isotropic and orthotropic material); elements for calculation multilayer shallow plates and shells, taking into account interlayer shifts and curvature; quadrangular and triangular plate elements on an elastic base; elements in the form of a tetrahedron, parallelepiped, general octahedron; special elements simulating links of finite stiffness, etc. A developed library of finite elements, effective methods and algorithms for solving equations systems of high order, and modern high-speed computers allow to solve problems with a large number of unknowns.

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